Bicuspidal Geodesics on Punctured Hyperbolic Surfaces

FYP Final Talk

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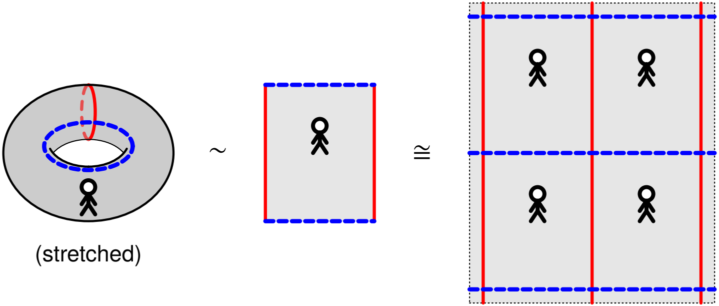
Geometry on surfaces

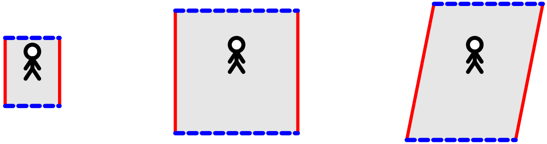
- Length and angle
  (= Riemannian metric)
- Straight lines
  (= Geodesics)
- Lengths of ________ geodesics
  - How many are there with length $\leq L$?
  - What can we deduce about the surface?

Example: flat torus

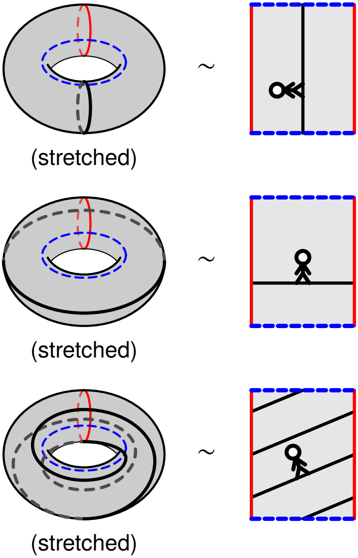
Locally isometric to Euclidean plane



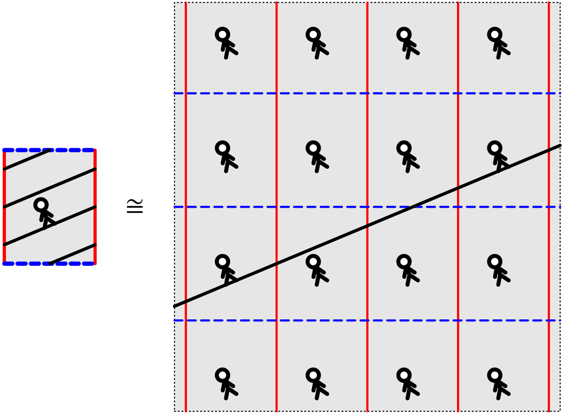
Some non-isometric flat tori:



Closed geodesics on flat tori

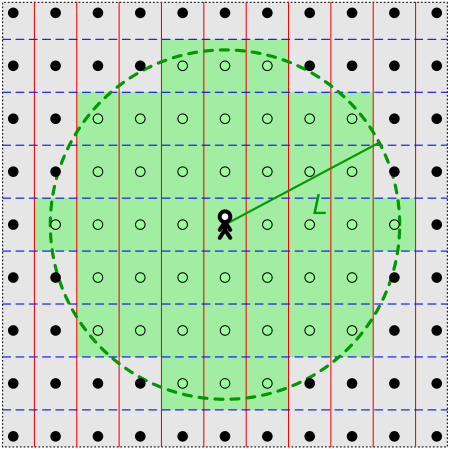


Universal covering = map



One image for each closed geodesic

Asymptotics for closed geodesics



$$\#\left\{\begin{array}{c}\text{closed geodesics}\\ \text{of length} \leq L\end{array}\right\} = \frac{\pi}{\text{Area}(S)}L^2 + O(L)$$

Application: number theory

How many ways are there to write integers up to 1000 as sums of two squares?

$$
\begin{aligned}
&\# \left\{(m,n) \in \mathbb{Z}^2 \,:\, m^2 + n^2 \le 1000\right\} \\
&= \# \left\{ \begin{matrix} \text{Closed geodesics in } S \\ \text{of length} \le \sqrt{1000} \end{matrix} \right\} \\
&\approx \frac{\pi}{\text{Area}(S)} L^2 \\
&\approx 1000\pi
\end{aligned}
$$

$$\left(S = \begin{matrix} \text{square torus} \\ \text{of area 1} \end{matrix} \right)$$

$$(L = \sqrt{1000})$$

Recap: closed geodesics on flat tori

- Lift to universal cover
- Find asymptotics using geometric reasoning
- Applications to number theory (for some surfaces)

Hyperbolic geometry = –(Spherical geometry)

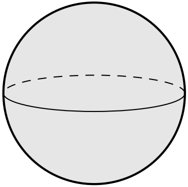
$K \equiv 0$

Euclidean plane

- Flat

$K \equiv 1$

Sphere



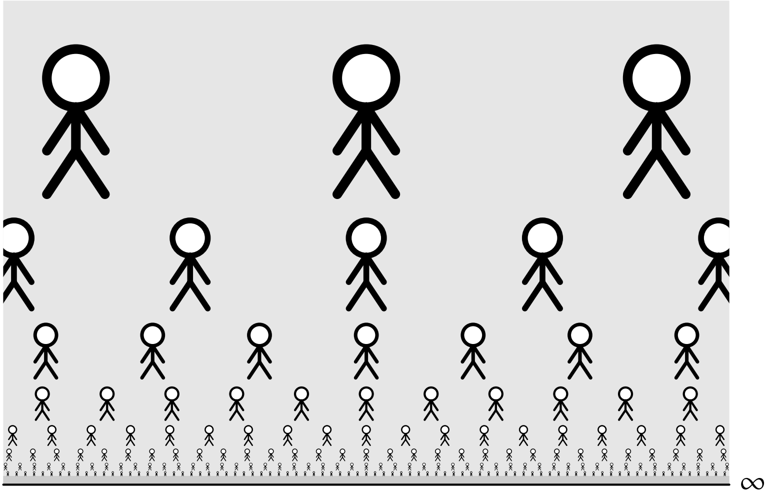
- Area deficit
- Geodesics converge

$K \equiv -1$

Hyperbolic plane

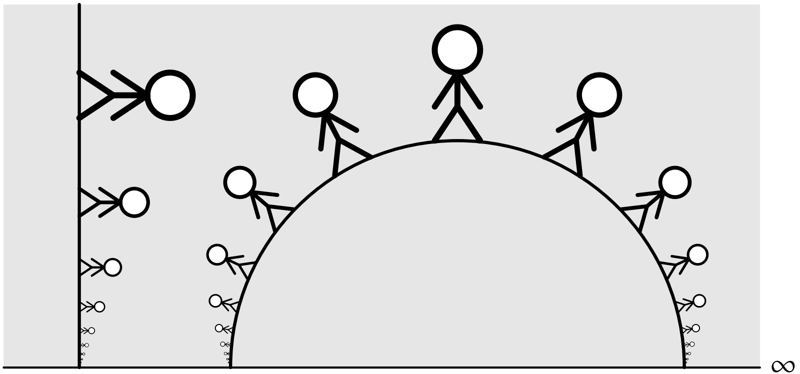
- Area excess
- Geodesics diverge

The hyperbolic plane



Poincaré half-plane model

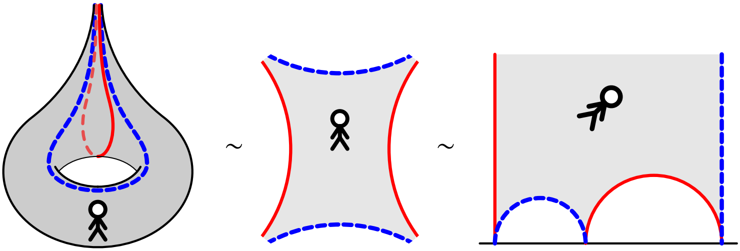
Geodesics on the hyperbolic plane



Hyperbolic surfaces

Gaussian curvature $\equiv -1$

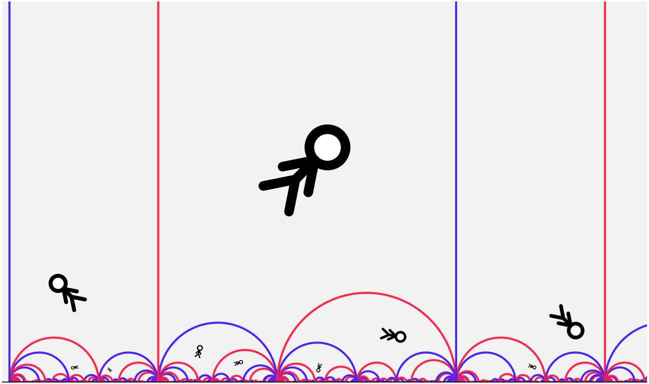
Locally isometric to hyperbolic plane



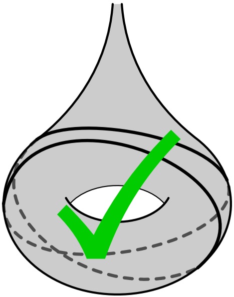
(stretched) (stretched)

Hyperbolic surfaces

$\cong$



Counting geodesics



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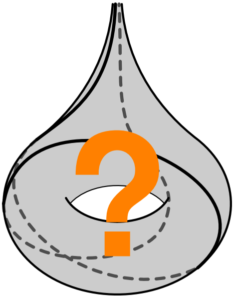
closed
geodesics

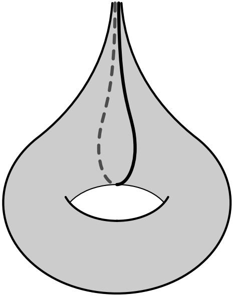


(stretched)

bicuspidal
geodesics

Asymptotics for bicuspidal geodesics

Which of these bicuspidal geodesics is longer?



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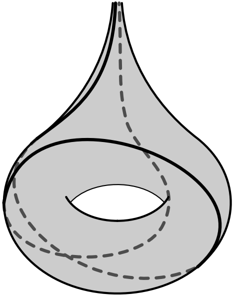

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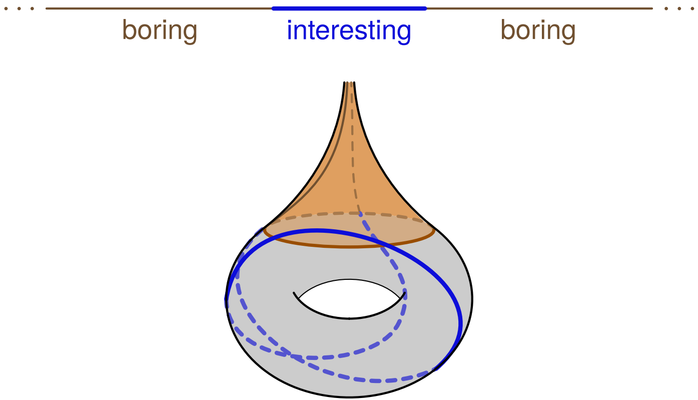
Answer: both have infinite length!
Need to normalise lengths

Cusp regions



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Definition

Normalised length = length of **interesting** part

Goal

Main question

Given two cusps C_1 and C_2 on a hyperbolic surface S, find asymptotics for $\#\left\{\begin{array}{c}\text{bicuspidal geodesics from } C_1 \text{ to } C_2\\ \text{with normalised length} \leq L\end{array}\right\}$.

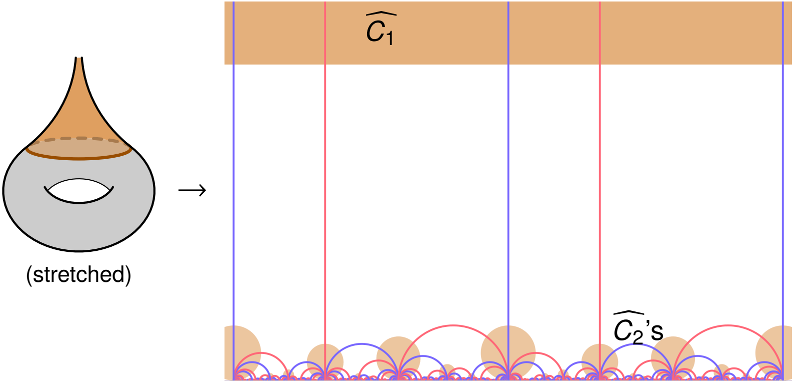
Guess: proportional to $\mathrm{Area}(\text{circle of radius } L)$

$$\#\left\{\begin{array}{c}\text{bicuspidal geodesics from } C_1 \text{ to } C_2\\ \text{with normalised length} \leq L\end{array}\right\} \overset{?}{\sim} c_S e^L$$

Plan of attack:

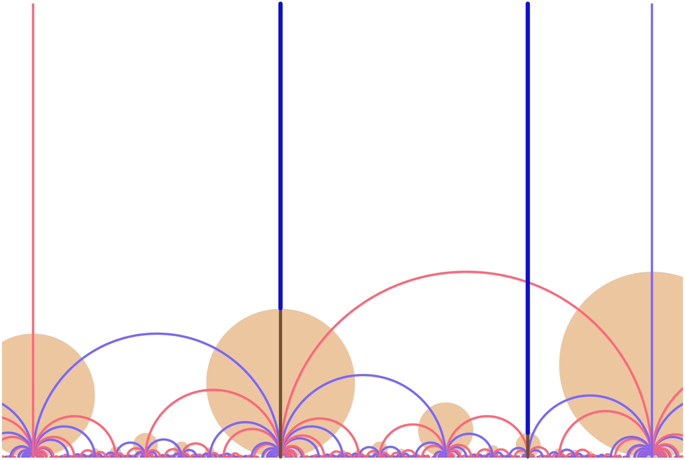
- Lift to universal cover
- Find asymptotics using geometric reasoning
- Applications to number theory (for some surfaces)

Step 1: Lift to universal cover



- Periodic: repeats horizontally
- Images of C_1 and C_2 are disjoint (Collar theorem)
- One image of C_2 for each bicuspidal geodesic

Step 1: Lift to universal cover



Normalised length ↔ size of image

Step 2: Geometric reasoning

Can't fit too many big circles in a vertical strip

$\rightsquigarrow$ Can't have too many short bicuspidal geodesics

Theorem

$$\#\left\{\begin{array}{c}\text{bicuspidal geodesics from } C_1 \text{ to } C_2\\ \text{with normalised length} \leq L\end{array}\right\} \leq 2e^L.$$

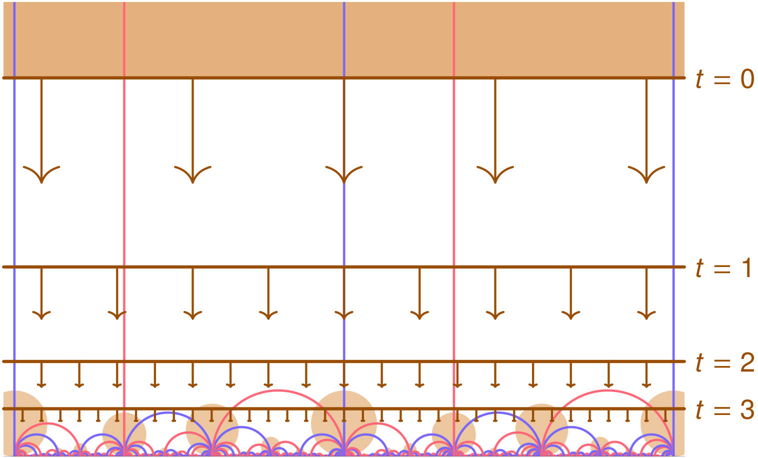
Pros:

- Elementary argument
- Works for any such arrangement of circles

Cons:

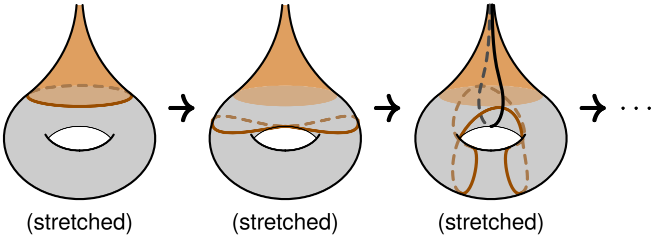
- Uses no information specific to hyperbolic surfaces
- Too weak to get correct constant

Step 2b: More geometric reasoning



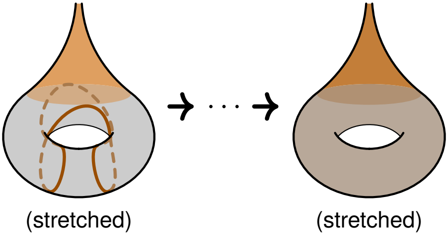
$$\#\left\{\begin{array}{c}\text{bicuspidal geodesics with}\\ \text{normalised length} \leq L\end{array}\right\} = \#\left\{\begin{array}{c}\text{wavefront entries to } C_2\\ \text{after time } L\end{array}\right\}$$

Wavefronts on hyperbolic surfaces



What happens in the long run?

Wavefronts on hyperbolic surfaces



Theorem (Zagier 1981, Sarnak 1981; Eskin-McMullen 1993)

Wavefronts from cusp regions equidistribute with respect to area as $t \longrightarrow \infty$.

$$\text{proportion of wavefront in } C_2 \longrightarrow \frac{\text{Area}(C_2)}{\text{Area}(S)} = \frac{2}{\text{Area}(S)}.$$

Wavefronts on hyperbolic surfaces

$$\#\left\{\begin{array}{c}\text{wavefront entries to } C_2\\ \text{after time } L\end{array}\right\} = \#\left\{\begin{array}{c}\text{bicuspidal geodesics with}\\ \text{normalised length at most } L\end{array}\right\}$$

$\updownarrow$ Technical estimates

$$\begin{array}{c}\text{proportion of}\\ \text{wavefront in } C_2\end{array} \longrightarrow \frac{2}{\mathrm{Area}(S)}$$

Step 2c: Technical estimates

$$\begin{matrix}\text{proportion of} \\ \text{wavefront in } C_2\end{matrix} \rightsquigarrow \# \left\{ \begin{matrix}\text{wavefront entries to } C_2 \\ \text{after time } L\end{matrix} \right\}$$

Using Wiener's tauberian theorem, we prove:

Main estimate

Let $f : [0, \infty) \longrightarrow \mathbb{R}$ be a function satisfying some technical conditions, and let $0 < \ell_1 \leq \ell_2 \leq \cdots$. If

$$\sum_{\ell_k \leq L} e^{-\ell_k} f(L - \ell_k) \longrightarrow \alpha \quad \text{as } L \longrightarrow \infty,$$

then

$$\#\{k : \ell_k \leq L\} \sim \frac{\alpha}{\int f} e^L \quad \text{as } L \longrightarrow \infty.$$

Main theorem

Theorem

$$\#\left\{\begin{array}{c}\text{bicuspidal geodesics from } C_1 \text{ to } C_2 \\ \text{with normalised length} \leq L\end{array}\right\} \sim \frac{8}{\pi \operatorname{Area}(S)} e^L.$$

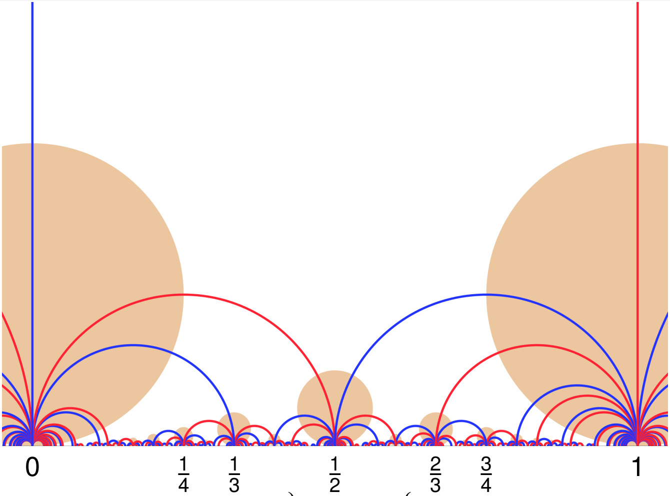
Length spectrum gives information about topology of S:

$$\operatorname{Area}(S) = 2(2g - 2 + p)\pi,$$

g = genus

p = number of cusps

Example: the modular torus



$$\#\left\{\begin{array}{c}\text{bicuspidal geodesics with}\\ \text{normalised length} \leq L\end{array}\right\} = 6\cdot\#\left\{\begin{array}{c}\text{rational numbers in } [0,1)\\ \text{with denominator} \leq n\end{array}\right\}$$

$$(L = 2\ln n + 2\ln 3)$$

Example: the modular torus

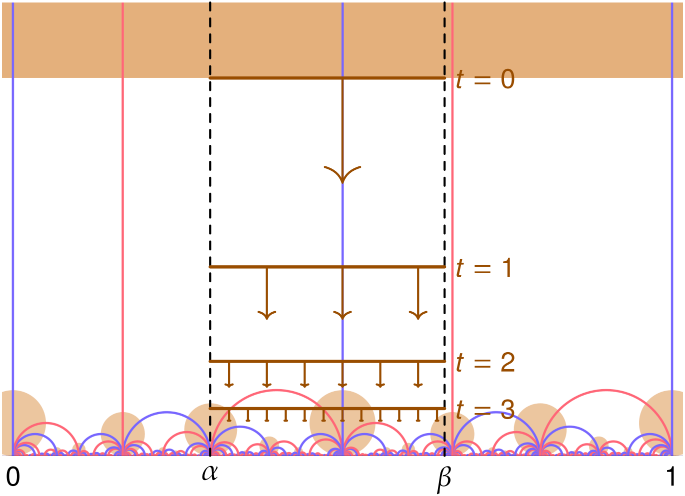
$$\# \left\{ \begin{array}{c} \text{bicuspidal geodesics with} \\ \text{normalised length} \leq L \end{array} \right\} \sim \frac{2}{\pi^2} e^L$$

$$\therefore \# \left\{ \begin{array}{c} \text{rational numbers in } [0,1) \\ \text{with denominator} \leq n \end{array} \right\} \sim \frac{3}{\pi^2} n^2$$

$$\sim \frac{1}{2\zeta(2)} n^2$$

Corollary

$$\zeta(2) = \frac{\pi^2}{6}.$$


t = 0
t = 1
t = 2
t = 3
0
α
β
1

Bonus: local asymptotics

Theorem (Hejhal 1996)

$[\alpha, \beta]$-segments of wavefronts equidistribute with respect to area as $t \longrightarrow \infty$.

Using the same machinery, we obtain:

Theorem

$$\#\left\{\begin{array}{c}\text{bicuspidal geodesics from } C_1 \text{ to } C_2\\ \text{lying over } [\alpha, \beta]\\ \text{with normalised length} \leq L\end{array}\right\} \sim \frac{8}{\pi\,\mathrm{Area}(S)}(\beta - \alpha)e^L.$$

Corollary

$\left\{\begin{array}{c}\text{bicuspidal geodesics from } C_1 \text{ to } C_2\\ \text{with normalised length} \leq L\end{array}\right\}$ equidistribute around C_1 as $L \longrightarrow \infty$.

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