The de Rham Cohomology

MA5209 Reading Report 1

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Algebraic topology is the study of algebraic objects derived from topological spaces, invariant under homeomorphisms or homotopy equivalence. Cohomology is a powerful family of such tools developed in the 20th century. In this report, we cover the fundamental concepts of de Rham cohomology for smooth manifolds.

The material presented here essentially follows §1, 2, 4, 5 of Bott and Tu (1982), and Ch. 4 §6 of Guillemin and Pollack (1974).

Differential forms

Let $x_1, \ldots, x_n$ denote the coordinates on $\mathbb{R}^n$. Recall that we may define the *exterior algebra* $\Omega^* = \bigwedge((\mathbb{R}^n)^*)$, the $\mathbb{R}$-algebra generated by $dx_1, \ldots, dx_n$ with relations $dx_i\, dx_j = -dx_j\, dx_i$. Note that

$$\Omega^* = \bigoplus_{q=0}^{\infty} \Omega^q, \quad \text{with} \quad \Omega^q = \operatorname{span}\{dx_{i_1} \cdots dx_{i_q} : 1 \leqslant i_1 < \cdots < i_q \leqslant n\}.$$

Since $\Omega^p \Omega^q \subseteq \Omega^{p+q}$, Ω^* is a graded algebra.

Let $U \subseteq \mathbb{R}^n$ be an open set. The *differential* q*-forms* on U are the elements of $\Omega^q(U) = C^\infty(U) \otimes_{\mathbb{R}} \Omega^q$, ie. any q-form can be uniquely written as

$$\sum_{1 \leqslant i_1 < \cdots < i_q \leqslant n} f_{i_1,\ldots,i_q}\, dx_{i_1} \cdots dx_{i_q} = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I| = q}} f_I\, dx_I, \qquad \begin{aligned} f_I &= f_{i_1,\ldots,i_q} \in C^\infty(U), \\ dx_I &= dx_{i_1} \cdots dx_{i_q}. \end{aligned}$$

This gives the graded algebra $\Omega^*(U) = \bigoplus_{q=1}^\infty \Omega^q(U)$ of differential forms on U.

(Equivalently, a q-form ω on U is a smooth cross-section of the q^{th} exterior power of the cotangent bundle, $\omega : U \to \bigwedge^k T^*U$.)

Exterior derivative The *exterior derivative* $d : \Omega^q(U) \to \Omega^{q+1}(U)$ is defined by:

- If $f \in \Omega^0(U) = C^\infty(U)$, then $df = \sum \frac{\partial f}{\partial x_i}\, dx_i$;
- If $\omega = \sum f_I\, dx_I$, then $d\omega = \sum df_i\, dx_I$.

Proposition. d *is an antiderivation, ie.* $d(\tau\omega) = (d\tau)\omega + (-1)^{\deg \tau}\tau(d\omega)$.

Proof. We check that this identity holds on monomials $\tau = f_I\, dx_I$, $\omega = g_J\, dx_J$. By the product rule,

$$\begin{aligned} d(\tau\omega) = d(f_I g_J)\, dx_I\, dx_J &= (df_I) g_J\, dx_I\, dx_J + f_I (dg_J)\, dx_I\, dx_J \\ &= (d\tau)\omega + (-1)^{\deg \tau}\tau(d\omega), \end{aligned}$$

since $dx_k\, dx_I = (-1)^{\deg\tau} dx_I\, dx_k$. The result follows by linearity. $\square$

Proposition. $d^2 = 0$.

Proof. For $f \in C^\infty(U)$, we have

$$d^2 f = d\left(\sum_i \frac{\partial f}{\partial x_i} dx_i\right) = \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j\, dx_i.$$

By equality of mixed partials ($\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}$), and the antisymmetry $dx_i\, dx_j = -dx_j\, dx_i$, all terms on the right cancel out. Hence $d^2 f = 0$.

For monomials $\omega = f_I\, dx_I$, we have $d^2\omega = d(df_I\, dx_I) = (d^2 f_I)\, dx_I - df_I\, d(dx_I)$. Now $d^2 f_I = 0$ by the above argument, and $d(dx_I) = 0$ by direct computation. Hence $d^2\omega = 0$, and the result follows by linearity. $\square$

Pullbacks Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be open sets. Then any smooth map $f : U \to V$ induces a pullback map $f^* : C^\infty(V) \to C^\infty(U)$, defined by $f^*(g) = g \circ f$. We may extend this to $f^* : \Omega^*(V) \to \Omega^*(U)$ by

$$f^*\left(\sum g_I\, dy_{i_1} \cdots dy_{i_q}\right) = \sum (g_I \circ f)\, df_{i_1} \cdots df_{i_q}.$$

Proposition. f^* *commutes with* d.

Proof. By linearity, we only need to check $f^* d = d f^*$ for monomials:

$$\begin{aligned} f^* d(g_I\, dy_{i_1} \cdots dy_{i_q}) &= f^*\left(\sum_i \frac{\partial g_I}{\partial y_i} dy_i\, dy_{i_1} \cdots dy_{i_q}\right) \\ &= \sum_i \left(\frac{\partial g_I}{\partial y_i} \circ f\right) df_i\, df_{i_1} \cdots df_{i_q} \\ &= d(g_I \circ f)\, df_{i_1} \cdots df_{i_q} \\ &= d((g_I \circ f)\, df_{i_1} \cdots df_{i_q}) = df^*(g_I\, dy_{i_1} \cdots dy_{i_q}). \end{aligned}$$

$\square$

Forms on smooth manifolds A *smooth manifold* M is a Hausdorff, second-countable topological space with an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ such that $\{U_\alpha\}$ is an open cover of M, $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$ are homeomorphisms, and the transition functions $g_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ are diffeomorphisms on $\mathbb{R}^n$.

The notion of a differential form carries over to smooth manifolds. In terms of an atlas, a q-form $\omega \in \Omega^q(M)$ on a smooth manifold M is a collection of q-forms $\omega_\alpha \in \Omega^q(\varphi_\alpha(U_\alpha))$, which are compatible in the sense that $\omega_\beta = g^*_{\alpha\beta}\omega_\alpha$ on $\varphi_\beta(U_\alpha \cap U_\beta)$ for all α, β. (Alternatively, we can define a q-form ω on M as a smooth cross-section of the q^{th} exterior power of the cotangent bundle, $\omega : M \to \wedge^k T^*M$.) The exterior derivative can then be defined by pullback:

$$(d\omega)|_{U_\alpha} = \varphi^*_\alpha(d\omega_\alpha).$$

Note that all propositions above carry over to the setting of smooth manifolds; in particular, any smooth map $f : M \to N$ between manifolds induces (via pullback to $\mathbb{R}^n$) a pullback map on forms $f^* : \Omega^*(N) \to \Omega^*(M)$, which commutes with the exterior derivative. Hence Ω^* can be seen as a contravariant functor on the category of smooth manifolds.

The de Rham cohomology

A differential form ω on a manifold M is called *closed* if $d\omega = 0$, and *exact* if $\omega = d\tau$ for some form τ. Since $d^2 = 0$, all exact forms are closed. Hence we may define the quotient vector space

$$\begin{aligned} H^q(M) &= \{\text{closed q-forms}\}/\{\text{exact q-forms}\} \\ &= (\ker d \cap \Omega^q(M))/(\operatorname{im} d \cap \Omega^q(M)). \end{aligned}$$

This is known as the q^{th} *de Rham cohomology* of M. We now list some basic properties and examples.

Basic properties For any $f \in C^\infty(M)$, $df = 0$ if and only if f is locally constant, ie. constant on each connected component. Hence $H^0(M) = \mathbb{R}^k$, where k is the number of connected components of M.

Also, $\Omega^q(M) = 0 \implies H^q(M) = 0$ for $q > \dim M$.

Example: $H^*(\mathbb{R}^1)$ For any 1-form $\omega = g(x)\,dx$ on $\mathbb{R}^1$, set $f(x) = \int_0^x g(u)\,du$. Then $df = \omega$, so every 1-form is exact. Hence $H^q(\mathbb{R}^1) = \begin{cases} \mathbb{R} & q = 0 \\ 0 & q \geqslant 1. \end{cases}$

Example: $H^*(S^1)$ For any exact 1-form df on $S^1 = \mathbb{R}/\mathbb{Z}$, we have $\int_{S^1} df = \int_0^1 df = f(1) - f(0) = 0$. Conversely, for any 1-form $\omega = g(x)\,dx$ on S^1 satisfying $\int_{S^1}\omega = 0$, set $f(x) = \int_0^x g(u)\,du$; then $df = \omega$. Hence $\operatorname{im} d \cap \Omega^1(S^1)$ is the kernel of the linear functional $\int_{S^1} : \Omega^1(S^1) \to \mathbb{R}$.

Now $\ker d \cap \Omega^1(S^1) = \Omega^1(S^1)$, so $H^q(S^1) = \begin{cases} \mathbb{R} & q = 0, 1 \\ 0 & q \geqslant 2 \end{cases}$ by the isomorphism theorem.

Differential complexes The structure of $\Omega^*(M)$ with exterior derivative d is captured by the notion of a differential complex (or cochain complex). In general, a direct sum of vector spaces $C = \bigoplus_{q \in \mathbb{Z}} C^q$ is a *differential complex* if there are homomorphisms

$$\cdots \to C^{q-1} \xrightarrow{d} C^q \xrightarrow{d} C^{q+1} \to \cdots$$

such that $d^2 = 0$. Then the *cohomology* of C is the direct sum of vector spaces $H^*(C) = \bigoplus_{q \in \mathbb{Z}} H^q(C)$, where

$$H^q(C) = \frac{\ker d \cap C^q}{\operatorname{im} d \cap C^q}.$$

A linear map $f : A \to B$ between differential complexes is a *chain map* if it respects the differential operator, ie. $f d_A = d_B f$. Note that chain maps induce a linear map between the cohomologies.

In particular, if $f : M \to N$ is a smooth map between manifolds, then the pullback $f^* : \Omega^*(N) \to \Omega^*(M)$ commutes with the exterior derivative, and thus is a chain map. This induces a linear map $f^\# : H^*(N) \to H^*(M)$ between the cohomologies. Note that if f is a diffeomorphism, then $f^\#$ is an isomorphism.

Proposition. *Given a short exact sequence of differential complexes*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0,$$

where f, g *are chain maps, there is a long exact sequence of cohomology groups*

$$\begin{tikzcd}
 & H^{q+1}(A) \arrow[r, "f^\#"] & \cdots \\
 & H^q(A) \arrow[r, "f^\#"] & H^q(B) \arrow[r, "g^\#"] & H^q(C) \arrow[ulll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=1ex]\tikztotarget.south) \tikztonodes -- ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}, "d^*" near end]
\end{tikzcd}$$

Proof. Firstly, we need to define d^*.

$$\begin{tikzcd}
 & \uparrow & \uparrow & \uparrow & \\
0 \arrow[r] & A^{q+1} \arrow[u] \arrow[r, "f"] & B^{q+1} \arrow[u] \arrow[r, "g"] & C^{q+1} \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & A^q \arrow[u] \arrow[r, "f"] & B^q \arrow[u, "d"] \arrow[r, "g"] & C^q \arrow[u, "d"] \arrow[r] & 0 \\
 & \arrow[u] & \arrow[u] & \arrow[u] &
\end{tikzcd}$$

Take $c \in \ker d \cap C^q$. By surjectivity of g, there exists $b \in B^q$ with $g(b) = c$. Now $dc = dg(b) = g(db) = 0$, so $db = f(a)$ for some $a \in A^{q+1}$ (since $\ker g = \operatorname{im} f$). Note that $f(da) = df(a) = d^2 b = 0$, so by injectivity of f we have $da = 0$, ie. $a \in \ker d \cap A^{q+1}$.

Moreover, for any other choice c', b', a' in the above construction, with $c - c' \in \operatorname{im} d$, we have

$$g(b - b') = c - c' = dc_0 = dg(b_0) = g(db_0),$$

so $b - b' - db_0 \in \ker g = \operatorname{im} f$ implies $b - b' - db_0 = f(a_0)$ for some $a_0 \in A^q$. Now

$$f(a - a') = db - db' = d(b - b' - db_0) = df(a_0) = f(da_0),$$

so by injectivity we have $a - a' = da_0 \in \operatorname{im} d$. Hence the expression

$$d^*(c + (\operatorname{im} d \cap C^q)) = a + (\operatorname{im} d \cap A^{q+1})$$

gives a well-defined linear map $d^* : H^q(C) \to H^{q+1}(A)$.

The fact that the sequence given above is exact can be verified in a routine manner, so we omit the details. $\square$

The Mayer-Vietoris sequence

Suppose that $M = U \cup V$, with U, V open. To relate the cohomology of M with the cohomology of U and V, consider the sequence of inclusions

$$M \longleftarrow U \coprod V \underset{\iota_V}{\overset{\iota_U}{\leftleftarrows}} U \cap V,$$

where $U \coprod V = U \times \{0\} \cup V \times \{1\}$ is the disjoint union, and ι_U, ι_V are inclusion maps. Then the contravariant functor Ω^* yields maps

$$\Omega^*(M) \longrightarrow \Omega^*(U) \oplus \Omega^*(V) \underset{\iota_V^*}{\overset{\iota_U^*}{\rightrightarrows}} \Omega^*(U \cap V),$$

each of which is a restriction of differential forms (ie. pullback induced by the inclusion). Taking the difference of the last two maps, we get the *Mayer-Vietoris sequence*

$$\begin{array}{ccccc} 0 \to \Omega^*(M) \xrightarrow{f} & \Omega^*(U) \oplus \Omega^*(V) & \xrightarrow{g} & \Omega^*(U \cap V) \to 0 \\ & (\omega, \tau) & \mapsto & \tau - \omega. \end{array}$$

Proposition. *The Mayer-Vietoris sequence is exact.*

Proof. Since $\omega \in \Omega^*(M)$ is uniquely determined by $\omega|_U$ and $\omega|_V$, f is injective.

Note that $(\omega, \tau) \in \ker g$ if and only if $\omega|_{U\cap V} = \tau|_{U\cap V}$. Every element of $\operatorname{im} f$ satisfies this condition, since $(\omega|_U)|_{U\cap V} = (\omega|_V)|_{U\cap V} = \omega|_{U\cap V}$; conversely, every (ω, τ) satisfying this condition can be glued together to give a smooth form on M. Hence $\ker g = \operatorname{im} f$.

Let $\{\rho_U, \rho_V\}$ be a partition of unity of M subordinate to the open cover $\{U, V\}$, ie. ρ_U, ρ_V are nonnegative C^∞ functions on M with $\operatorname{supp}(\rho_U) \subseteq U$, $\operatorname{supp}(\rho_V) \subseteq V$, and $\rho_U + \rho_V = 1$. Now for any $\omega \in \Omega^*(U \cap V)$, we have $\rho_V \omega \in \Omega^*(U)$ and $\rho_U \omega \in \Omega^*(V)$. Hence $g(-\rho_V \omega, \rho_U \omega) = \omega$, so g is surjective. $\square$

Example: $H^*(S^1)$ We recompute the cohomology of S^1, now using the Mayer-Vietoris sequence. Take the open cover of S^1 by two open intervals, say $S^1 = U \cup V$.

Note that $U \cap V$ is the union of two disjoint open intervals. Since open intervals are diffeomorphic to $\mathbb{R}^1$, the Mayer-Vietoris sequence for S^1 is as follows:

$$\begin{tikzcd}
 & S^1 & U \coprod V & U \cap V \\
H^1 & ? \arrow[r] & 0 \arrow[r] & 0 \\
H^0 & ? \arrow[r] & \mathbb{R} \oplus \mathbb{R} \arrow[r, "\delta"] & \mathbb{R} \oplus \mathbb{R} \arrow[ull, "d^*"']
\end{tikzcd}$$

Now for $(\omega, \tau) \in H^0(U \coprod V)$ (ie. ω, τ are constant functions on U, V respectively), the difference $\tau - \omega$ is constant on $U \cap V$. Hence the difference map δ takes (ω, τ) to $(\tau - \omega, \tau - \omega)$, so $\dim \operatorname{im} \delta = 1$ implies $\dim \ker \delta = 1$. Hence

$$\begin{aligned}
H^0(S^1) &\cong \ker \delta \cong \mathbb{R}, \\
H^1(S^1) &\cong \operatorname{im} d^* \cong \frac{\mathbb{R} \oplus \mathbb{R}}{\ker d^*} \cong \frac{\mathbb{R} \oplus \mathbb{R}}{\operatorname{im} \delta} \cong \mathbb{R}.
\end{aligned}$$

The Poincaré lemma

We now compute the cohomology $H^*(\mathbb{R}^n)$, which will turn out to have important consequences for how the cohomology behaves under homotopy.

Let $\pi : \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^n$ and $s : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^1$ be the canonical projection and inclusion maps, respectively:

$$\begin{aligned} \pi &: (x, t) \mapsto x \\ s &: x \mapsto (x, 0) \end{aligned} \qquad \begin{tikzcd} \mathbb{R}^n \times \mathbb{R}^1 \arrow[d, shift left, "\pi"] \\ \mathbb{R}^n \arrow[u, shift left, "s"] \end{tikzcd} \qquad \begin{tikzcd} \Omega^*(\mathbb{R}^n \times \mathbb{R}^1) \arrow[d, shift right, "s^*"'] \\ \Omega^*(\mathbb{R}^n) \arrow[u, shift right, "\pi^*"'] \end{tikzcd}$$

Theorem. *The induced maps* $H^*(\mathbb{R}^n) \underset{s^\#}{\overset{\pi^\#}{\rightleftarrows}} H^*(\mathbb{R}^n \times \mathbb{R}^1)$ *are isomorphisms, inverse to each other.*

Proof. Since $\pi \circ s = \mathbb{1}$ on $\mathbb{R}^n$, we have $s^* \circ \pi^* = \mathbb{1}$ on $\Omega^*(\mathbb{R}^n)$, so $s^\# \circ \pi^\# = \mathbb{1}$ on $H^*(\mathbb{R}^n)$. It remains to show that $\pi^\# \circ s^\# = \mathbb{1}$ on $H^*(\mathbb{R}^n \times \mathbb{R}^1)$.

Note that any q-form on $\mathbb{R}^n \times \mathbb{R}^1$ is a linear combination of forms of the following two types:

$$\begin{aligned}
&(\mathrm{I})\ f\,\pi^*\varphi, && \varphi = dx_I \in \Omega^q(\mathbb{R}^n), \quad f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^1);\ \text{or} \\
&(\mathrm{II})\ f\,\pi^*\psi\, dt, && \psi = dx_J \in \Omega^{q-1}(\mathbb{R}^n), \quad f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^1).
\end{aligned}$$

Consider the linear map K on $\Omega^*(\mathbb{R}^n \times \mathbb{R}^1)$ that sends forms of type (I) to 0, and forms of type (II) to $(\int_0^t f)\,\pi^*\psi$. We now claim that

$$\mathbb{1} - \pi^* \circ s^* = (-1)^{q-1}(dK - Kd) \qquad \text{on } \Omega^*(\mathbb{R}^n \times \mathbb{R}^1). \tag{*}$$

Then $\mathbb{1} - \pi^* \circ s^*$ maps closed forms to exact forms, which descend to 0 in cohomology, so $\pi^\# \circ s^\# = \mathbb{1}^\# = \mathbb{1}$.

We now check (*) for forms of type (I) and (II) respectively:

$$
\begin{aligned}
(\mathrm{I}): \quad (dK - Kd)(f\,\pi^*\varphi) &= -Kd(f\,\pi^*\varphi)\\
&= -K\left(\left(\sum_i \frac{\partial f}{\partial x_i}\,dx_i\, + \frac{\partial f}{\partial t}\,dt\right)\pi^*\varphi + f\,d(\pi^*\varphi)\right)\\
&= -K\left((-1)^q\frac{\partial f}{\partial t}\,\pi^*\varphi\;dt\right)\\
&= (-1)^{q-1}\int_0^t \frac{\partial f}{\partial t}\,\pi^*\varphi\\
&= (-1)^{q-1}(f(x,t) - f(x,0))\,\pi^*\varphi\\
&= (-1)^{q-1}(\mathbb{1} - \pi^*\circ s^*)(f\,\pi^*\varphi).
\end{aligned}
$$

$$
\begin{aligned}
(\mathrm{II}): \quad dK(f\,\pi^*\psi\;dt) &= d\left(\left(\int_0^t f\right)\pi^*\psi\right)\\
&= \left(\sum_i \int_0^t \frac{\partial f}{\partial x_i}\,dx_i\, + f\,dt\right)\pi^*\psi,\\
Kd(f\,\pi^*\psi\;dt) &= K\left(\left(\sum_i \frac{\partial f}{\partial x_i}\,dx_i\, + \frac{\partial f}{\partial t}\,dt\right)\pi^*\psi\;dt\right)\\
&= \sum_i \int_0^t \frac{\partial f}{\partial x_i}\,dx_i\,\pi^*\psi\\
\therefore\ (dK - Kd)(f\,\pi^*\psi\;dt) &= (-1)^{q-1}f\,\pi^*\psi\;dt\\
&= (-1)^{q-1}(\mathbb{1} - \pi^*\circ s^*)(f\,\pi^*\psi\;dt),
\end{aligned}
$$

since $s^*(dt) = d(s^*t) = d(0) = 0$. By linearity, (*) holds on all q-forms, as desired.□

Hence $H^*(\mathbb{R}^n) = H^*(\mathbb{R}^{n+1})$, so by induction we have:

Corollary (Poincaré lemma). $H^q(\mathbb{R}^n) = H^*(\textit{point}) = \begin{cases} \mathbb{R} & q = 0 \\ 0 & q \geqslant 1. \end{cases}$

More generally, if M is a smooth manifold then we can consider the canonical projection and inclusion maps $\pi : M \times \mathbb{R}^1 \to M$ and $s : M \to M \times \mathbb{R}^1$. Any atlas $\{(U_\alpha, \varphi_\alpha)\}$ for M gives an atlas $\{(U_\alpha \times \mathbb{R}^1, \varphi_\alpha \times \mathbb{1})\}$ for $M \times \mathbb{R}^1$. By pulling back to each chart and repeating the argument in the proof of the theorem, we obtain:

Corollary. *The induced maps* $H^*(M) \underset{s^\#}{\overset{\pi^\#}{\rightleftarrows}} H^*(M \times \mathbb{R}^1)$ *are isomorphisms, inverse to each other.*

Example: $H^*(S^n)$ Assume that $H^q(S^{n-1}) = \begin{cases} \mathbb{R} & q = 0, n-1 \\ 0 & q \neq 0, n-1 \end{cases}$ for some $n \geqslant 2$. (This is true for $n = 2$, from the computation of $H^*(S^1)$.) Take two points $P, P' \in S^n$, and consider the open cover $S^n = U \cup V$, with $U = S^n \backslash \{P\}$, $V = S^n \backslash \{P'\}$. Then $U \simeq V \simeq \mathbb{R}^n$, and $U \cap V \simeq \mathbb{R}^n \backslash \{0\} \simeq S^{n-1} \times \mathbb{R}^1$. Hence we have the following Mayer-Vietoris sequence for S^n:

$$\begin{tikzcd}
 & S^n & U \coprod V & U \cap V \\
H^n & ? \arrow[r] & 0 \arrow[r] & 0 \\
H^{n-1} & ? \arrow[r] & 0 \arrow[r] & \mathbb{R} \arrow[ull, "d^*"'] \\
\vdots & & \cdots \arrow[ul] & \\
H^1 & ? \arrow[r] & 0 \arrow[r] & 0 \arrow[ul] \\
H^0 & ? \arrow[r] & \mathbb{R} \oplus \mathbb{R} \arrow[r, "\delta"] & \mathbb{R} \arrow[ull, "d^*"']
\end{tikzcd}$$

Now the difference map δ takes (ω, τ) to $\tau - \omega$, so $\dim \operatorname{im} \delta = 1$ implies $\dim \ker \delta = 1$. Hence

$$\begin{aligned}
H^0(S^n) &\cong \ker \delta \cong \mathbb{R}, \\
H^1(S^n) &\cong \operatorname{im} d^* \cong \frac{\mathbb{R}}{\ker d^*} \cong \frac{\mathbb{R}}{\operatorname{im} \delta} \cong 0, \\
H^q(S^n) &\overset{d^*}{\cong} H^{q-1}(U \cap V) = \begin{cases} \mathbb{R} & q = n \\ 0 & q \neq n \end{cases} \quad \text{for } q \geqslant 2.
\end{aligned}$$

Thus $H^q(S^n) = \begin{cases} \mathbb{R} & q = 0, n \\ 0 & q \neq 0, n. \end{cases}$ By induction, this holds for all $n \geqslant 2$.

Homotopy invariance

Let M, N be smooth manifolds. A *smooth homotopy* between two C^∞ maps $f, g : M \to N$ is a C^∞ map $F : M \times \mathbb{R}^1 \to N$ such that $\begin{cases} F(x,t) = f(x) & t \geqslant 1 \\ F(x,t) = g(x) & t \leqslant 0. \end{cases}$ By the Whitney embedding theorem, every continuous map between two manifolds is continuously homotopic to a C^∞ map (cf. Bott/Tu Proposition 17.8); hence two maps are smoothly homotopic if and only if they are (continuously) homotopic.

Proposition. *If* $f, g : M \to N$ *are homotopic, then the induced maps* $f^\#, g^\# : H^*(N) \to H^*(M)$ *are equal.*

Proof. Taking F as above, let $s_0, s_1 : M \to M \times \mathbb{R}^1$ be the 0- and 1-sections respectively: $s_0(x) = (x, 0)$, and $s_1(x) = (x, 1)$. Then

$$\begin{aligned}
f^\# &= (F \circ s_1)^\# = s_1^\# \circ F^\#, \\
g^\# &= (F \circ s_0)^\# = s_0^\# \circ F^\#.
\end{aligned}$$

But $s_0^\#$ and $s_1^\#$ are both inverses of $\pi^\#$, so they are equal. Hence $f^\# = g^\#$. $\square$

We say that M, N have the same *smooth homotopy type* if there are C^∞ maps $f : M \to N$ and $g : N \to M$ such that $g \circ f$ is homotopic to $\mathbb{1}_M$ and $f \circ g$ is homotopic to $\mathbb{1}_N$. By Proposition 17.8, this is equivalent to having the same homotopy type in the usual (continuous) sense.

Corollary. *If M, N have the same homotopy type, then $H^*(M) \cong H^*(N)$.*

Hence the de Rham cohomology gives us a useful way to distinguish between homotopy classes of manifolds.

Before proceeding, we recall from differential geometry that a *Riemannian metric* on M is a family of inner products $\langle \cdot, \cdot \rangle_p$ on T_pM for each $p \in M$, smooth in the sense that if X, Y are C^∞ vector fields on M then $\langle X, Y \rangle$ is a C^∞ function on M.

Lemma. *Every smooth manifold M admits a Riemannian metric.*

Proof. For an atlas $\{(U_\alpha, \varphi_\alpha)\}$ of M, let $\langle \cdot, \cdot \rangle_\alpha$ on U_α be the pullback of the standard inner product on $\varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$.

Now take a partition of unity $\{\rho_\alpha\}$ subordinate to the open cover $\{U_\alpha\}$. Then $\langle \cdot, \cdot \rangle = \sum_\alpha \rho_\alpha \langle \cdot, \cdot \rangle_\alpha$ is a well-defined Riemannian metric on M. $\square$

Recall that M is *contractible* if $\mathbb{1}_M$ is homotopic to some constant map, ie. M is homotopy equivalent to a point.

Proposition. *If M is a compact orientable manifold without boundary, then M is not contractible.*

Proof. Take a Riemannian metric $\langle \cdot, \cdot \rangle$ on M, which gives a volume form ω on M. Note that ω is closed since it has maximal degree. Also, ω is not exact, since if $\omega = d\tau$ then $0 = \int_{\partial M} \tau = \int_M \omega = \mathrm{Vol}(M) > 0$, contradiction. Hence $H^n(M) \neq 0 = H^n(\text{point})$, so M is not homotopy equivalent to a point. $\square$

The above statement also holds on non-orientable compact manifolds, by passing to the orientable double cover of M.

Homotopy invariance can also be used to give a cohomological proof of a classic theorem in algebraic topology.

Theorem (Brouwer fixed point theorem). *Let D^n be the closed unit ball in $\mathbb{R}^n$. Then every continuous map $f : D^n \to D^n$ has a fixed point, ie. there exists $x_0 \in D^n$ with $f(x_0) = f(x_0)$.*

Proof. Assume that there exists a continuous $f : D^n \to D^n$ with no fixed points. Then for each $x \in D^n$, the ray from $f(x)$ passing through x intersects $\partial D^n = S^{n-1}$ at a unique point, say $g(x)$. Then $g : D^n \to S^{n-1}$ is a continuous map.

Now if $i : S^{n-1} \hookrightarrow D^n$ is the inclusion map, then $g \circ i = \mathbb{1}_{S^{n-1}}$ (since $g(x) = x$ for $x \in S^{n-1}$). Moreover, the continuous map $F : D^n \times [0, 1] \to D^n$ defined by $F(x, t) = tx + (1 - t)g(x)$ shows that $g = i \circ g$ is homotopic to $\mathbb{1}_{D^n}$.

Thus D^n and S^{n-1} have the same homotopy type. But D^n is contractible implies $H^{n-1}(D^n) = 0 \not\cong \mathbb{R} = H^{n-1}(S^{n-1})$, which is the desired contradiction. $\square$

Finite dimensionality

The last result we shall prove is a criterion for the de Rham cohomology to be finite dimensional.

If $\dim M = n$, an open cover $\{U_\alpha\}$ of M is called a *good cover* if all nonempty finite intersections $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ are diffeomorphic to $\mathbb{R}^n$.

Proposition. *Every manifold M has a good cover. Moreover, if M is compact, then it has a finite good cover.*

Proof. Take a Riemannian metric $\langle \cdot, \cdot \rangle$ on M. Recall that every geodesically convex neighbourhood is diffeomorphic to $\mathbb{R}^n$, and the intersection of two geodesically convex sets is also geodesically convex. Hence any open cover of geodesically convex neighbourhoods about every point on M is a good cover. $\square$

Proposition. *If M has a finite good cover, then $H^*(M)$ is finite dimensional.*

Proof. Consider the Mayer-Vietoris sequence

$$\cdots \to H^{q-1}(U \cap V) \xrightarrow{d^*} H^q(U \cup V) \xrightarrow{r} H^q(U) \oplus H^q(V) \to \cdots,$$

which gives $H^q(U \cup V) \cong \ker r \oplus \operatorname{im} r \cong \operatorname{im} d^* \oplus \operatorname{im} r$. Hence if $H^q(U)$, $H^q(V)$ and $H^{q-1}(U \cap V)$ are finite dimensional, then so is $H^q(U \cup V)$.

Now suppose that any manifold with a good cover of cardinality at most p has finite dimensional cohomology (this is true for $p = 1$, by the Poincaré lemma), and M has a good cover $\{U_0, \ldots, U_p\}$ of size $p+1$. Let $U = U_0 \cup \cdots \cup U_{p-1}$ and $V = U_p$; then $U \cap V$ has a good cover $\{U_0 \cap U_p, \ldots, U_{p-1} \cap U_p\}$ of size p, so the cohomologies of U, V and $U \cap V$ are finite dimensional. Thus the cohomology of $M = U \cup V$ is also finite dimensional, and we are done by induction. $\square$

Conclusion

We end by sketching some connections between the de Rham theory and other theories of cohomology.

Historically, algebraic topology was first concerned with *homology*, the study of cycles and boundaries on simplicial complexes (*simplicial homology*), or more generally any topological space (*singular homology*). Taking the dual construction gives the *singular cohomology*, a topological invariant. By a theorem of de Rham, the de Rham cohomology of a manifold is naturally isomorphic to its singular cohomology; this implies that the de Rham cohomology is not just invariant under diffeomorphisms, but also under homeomorphisms!

Moreover, the de Rham cohomology has the following *computability* property: given a finite good cover $\{U_\alpha\}$ of M, the cohomology of M is completely determined by combinatorial data, namely which of the intersections $U_{\alpha_1} \cap \cdots \cap U_{\alpha_k}$ are nonempty. This can be deduced from a systematic study of the relation between de Rham cohomology and *Čech cohomology*, which is based on abstract simplicial complexes constructed from an open cover of a topological space.

The agreement between the de Rham cohomology and other cohomology theories is not merely coincidental. In the 1940s, Eilenberg and Steenrod gave a unified approach to the various cohomology theories, under the *Eilenberg-Steenrod axioms*. They showed that any theory satisfying these axioms, which includes all the cohomology theories mentioned above, agree at least on all CW complexes. Hence any such cohomolgy theory can be used interchangeably, depending on which is most convenient given the context (simplicial cohomology for simplicial complexes, de Rham cohomology for smooth manifolds, etc.).

References

R. Bott, L. W. Tu, *Differential Forms in Algebraic Topology*, Springer-Verlag, 1982.
V. Guillemin, A. Pollack, *Differential Topology*, Prentice-Hall, 1974.