The Čech-de Rham Complex

MA5209 Reading Report 2

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In the previous reading report, we covered the fundamental concepts of de Rham cohomology for smooth manifolds, and the Mayer-Vietoris sequence. In this report, we will study the generalisation to the Čech-de Rham differential complex

The material presented here essentially follows §8 and 9 of Bott and Tu (1982).

The Mayer-Vietoris sequence

We start by filling in details in the outline of proof given in the previous reading report, that the Mayer-Vietoris sequence induces a long exact sequence in cohomology.

Proposition (Snake lemma). *Consider the following commutative diagram in the category of vector spaces,*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & & \\ & & {\scriptstyle a}\uparrow & & {\scriptstyle b}\uparrow & & {\scriptstyle c}\uparrow & & \\ & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

where the rows are exact. Then there exists a homomorphism d *such that*

$$\ker a \xrightarrow{\tilde{f}} \ker b \xrightarrow{\tilde{g}} \ker c \xrightarrow{d} \operatorname{coker} a \xrightarrow{\tilde{f}'} \operatorname{coker} b \xrightarrow{\tilde{g}'} \operatorname{coker} c$$

is exact.

Proof. For $\gamma \in \ker c$, by surjectivity of g there exists $\beta \in B$ with $g(\beta) = \gamma$. Write $\beta' = b(\beta)$; then $g'(\beta') = c(g(\beta)) = c(\gamma) = 0$, so $\beta' \in \ker g' = \operatorname{im} f'$ implies there exists $\alpha' \in A'$ with $f'(\alpha') = \beta'$.

Moreover, for any other choice $\widehat{\beta} \in B$, $\widehat{\beta'} \in B'$, $\widehat{\alpha'} \in A'$ satisfying the above construction, we have $\widehat{\beta} - \beta \in \ker g = \operatorname{im} f$, so there exists $\varepsilon \in A$ with $f(\varepsilon) = \widehat{\beta} - \beta$. Hence

$$f'(a(\varepsilon)) = b(f(\varepsilon)) = \widehat{\beta'} - \beta' = f'(\widehat{\alpha'} - \alpha'),$$

so by injectivity of f' we have $\widehat{\alpha'} - \alpha' = a(\varepsilon) \in \operatorname{im} a$.

Thus for all $\gamma \in \ker c$, we may uniquely define $d(\gamma) = \alpha' + \operatorname{im} a \in \operatorname{coker} a$.

We now show exactness in four steps. Note that, by injectivity of f' $(*)$,

$$\operatorname{im} \tilde{f} \stackrel{*}{=} \operatorname{im} f \cap \ker b = \ker g \cap \ker b = \ker \tilde{g},$$

so $\ker a \xrightarrow{\tilde{f}} \ker b \xrightarrow{\tilde{g}} \ker c$ is exact. Similarly, by surjectivity of g $(**)$,

$$\operatorname{im} \tilde{f}' = \operatorname{im} f' / \operatorname{im} b = \ker g' / \operatorname{im} b \stackrel{**}{=} \ker \tilde{g}',$$

so $\operatorname{coker} a \xrightarrow{\tilde{f}'} \operatorname{coker} b \xrightarrow{\tilde{g}'} \operatorname{coker} c$ is exact.

If $\gamma \in \operatorname{im} \tilde{g}$, ie. $\gamma = g(\beta)$ with $\beta \in \ker b$, then in the above construction, $\beta' = b(\beta) = 0$, so by injectivity of f' we have $\alpha' = 0$, ie. $d(\gamma) = 0$. Conversely, if $d(\gamma) = 0$ then $\alpha' \in \operatorname{im} a$, say $\alpha' = a(\alpha)$. Then $b(\beta) = \beta' = f'(a(\alpha)) = b(f(\alpha))$, so $\beta - f(\alpha) \in \ker b$, and $g \circ f = 0$ implies $g(\beta - f(\alpha)) = g(\beta) = \gamma$. Hence $\gamma \in \operatorname{im} \tilde{g}$. Thus $\operatorname{im} \tilde{g} = \ker d$, so $\ker b \xrightarrow{\tilde{g}} \ker c \xrightarrow{d} \operatorname{coker} a$ is exact.

If $\alpha' + \operatorname{im} a \in \operatorname{im} d$, then in the above construction, $f'(\alpha') = \beta' = b(\beta) \in \operatorname{im} b$, so $\tilde{f}'(\alpha' + \operatorname{im} a) = 0 + \operatorname{im} b$. Conversely, if $\tilde{f}'(\alpha' + \operatorname{im} a) = 0 + \operatorname{im} b$, ie. $f'(\alpha') \in \operatorname{im} b$, say $f'(\alpha') = \beta' = b(\beta)$, then $\gamma = g(\beta)$ satisfies $d(\gamma) = \alpha' + \operatorname{im} a$. Hence $\alpha' + \operatorname{im} a \in \operatorname{im} d$. Thus $\operatorname{im} d = \ker \tilde{f}'$, so $\ker c \xrightarrow{d} \operatorname{coker} a \xrightarrow{\tilde{f}'} \operatorname{coker} b$ is exact. □

Recall that a *differential complex* (or *cochain complex*) is a direct sum of vector spaces $C = \oplus_{q \in \mathbb{Z}} C^q$ equipped with homomorphisms $d = d_q : C^q \to C^{q+1}$ such that $d_q \circ d_{q-1} = 0$. A *chain map* between differential complexes is a linear map $f : A \to B$ with $f \circ d^A = d^B \circ f$. The *cohomology* of C is $H^*(C) = \oplus_{q \in \mathbb{Z}} H^q(C)$, with $H^q(C) = \ker d_q / \operatorname{im} d_{q-1}$.

Corollary. *Given a short exact sequence of differential complexes*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \ ,$$

where f, g *are chain maps, there is a long exact sequence of cohomology groups*

$$\cdots \longrightarrow H^q(A) \xrightarrow{f^*} H^q(B) \xrightarrow{g^*} H^q(C) \xrightarrow{d^*} H^{q+1}(A) \xrightarrow{f^*} \cdots$$

Proof. We apply the snake lemma to the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker d^A_{q+1} & \xrightarrow{f} & \ker d^B_{q+1} & \xrightarrow{g} & \ker d^C_{q+1} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle d^A_q} & & \uparrow{\scriptstyle d^B_q} & & \uparrow{\scriptstyle d^C_q} & & \\
0 & \longrightarrow & \dfrac{A^q}{\operatorname{im} d^A_{q-1}} & \xrightarrow{f} & \dfrac{B^q}{\operatorname{im} d^B_{q-1}} & \xrightarrow{g} & \dfrac{C^q}{\operatorname{im} d^C_{q-1}} & \longrightarrow & 0
\end{array}$$

This gives an exact sequence

$$H^q(A) \xrightarrow{f^*} H^q(B) \xrightarrow{g^*} H^q(C) \xrightarrow{d^*} H^{q+1}(A) \xrightarrow{f^*} H^{q+1}(B) \xrightarrow{g^*} H^{q+1}(C)$$

Piecing together these exact sequences gives the desired long exact sequence. □

Let $\Omega^q(M)$ denote the differential q-forms on a smooth manifold M.

Suppose that $M = U \cup V$, with U, V open. Recall that we have the sequence of inclusions

$$M \longleftarrow U \coprod V \underset{\iota_V}{\overset{\iota_U}{\leftleftarrows}} U \cap V \ ,$$

where $U \coprod V = U \times \{0\} \cup V \times \{1\}$ is the disjoint union, and ι_U, ι_V are inclusion maps. Under the contravariant functor Ω^*, this induces maps

$$\Omega^*(M) \xrightarrow{r} \Omega^*(U) \oplus \Omega^*(V) \underset{\iota_V^*}{\overset{\iota_U^*}{\rightrightarrows}} \Omega^*(U \cap V) \ ,$$

each of which is a restriction of differential forms (ie. pullback induced by the inclusion). Taking the difference of the last two maps, we get the *Mayer-Vietoris sequence*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^*(M) & \xrightarrow{r} & \Omega^*(U) \oplus \Omega^*(V) & \xrightarrow{\delta} & \Omega^*(U \cap V) & \longrightarrow & 0 \\ & & & & (\omega, \tau) & \longmapsto & \tau - \omega & & \end{array}$$

Previously we checked that this sequence is exact, by using a partition of unity.

To anticipate the construction in the next section, we arrange the objects involved in the Mayer-Vietoris sequence the following table:

$$\begin{array}{c|ccc} q & & & \\ 3 & \vdots & \vdots & \vdots \\ 2 & \Omega^2(U) \oplus \Omega^2(V) & \Omega^2(U \cap V) & 0 \\ 1 & \Omega^1(U) \oplus \Omega^1(V) & \Omega^1(U \cap V) & 0 \\ 0 & \Omega^0(U) \oplus \Omega^0(V) & \Omega^0(U \cap V) & 0 \\ \hline & 0 & 1 & 2 \quad p \end{array}$$

Write $K^{p,q}$ for the (p, q)-entry of the table, so $K^{p,q} = \begin{cases} \Omega^q(U) \oplus \Omega^q(V) & p = 0 \\ \Omega^q(U \cap V) & p = 1 \\ 0 & p \geqslant 2. \end{cases}$

There are two differential operators naturally associated to this table, namely the exterior derivative d (going up each column) and the difference operator δ (going across each row), with $d^2 = 0$ and $\delta^2 = 0$ (since every δ other than the first is 0). The rows of the table are exact by the Mayer-Vietoris sequence, and the columns are exact only when U, V and $U \cap V$ have trivial cohomology, ie. $H^*(U) = H^*(V) = H^*(U \cap V) = H^*(pt)$.

Generalised Mayer-Vietoris sequence

More generally, instead of an open cover of M with two open sets $\{U, V\}$, we may consider an open cover $\mathfrak{U} = \{U_\alpha \ : \ \alpha \in J\}$, where J is a countable, totally ordered set. Denoting the finite intersection $U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_k}$ by $U_{\alpha_0\alpha_1\ldots\alpha_k}$, we have a sequence of inclusions

$$M \longleftarrow \coprod U_{\alpha_0} \underset{\partial_1}{\overset{\partial_0}{\leftleftarrows}} \coprod_{\alpha_0<\alpha_1} U_{\alpha_0\alpha_1} \overset{\partial_0}{\underset{\partial_2}{\overset{\partial_1}{\Lleftarrow}}} \coprod_{\alpha_0<\alpha_1<\alpha_2} U_{\alpha_0\alpha_1\alpha_2} \;\; \Lleftarrow \cdots$$

where ∂_i is the inclusion ignoring the ith open set, eg. for $\alpha < \beta < \gamma$,

$$\partial_0 : U_{\alpha\beta\gamma} \hookrightarrow U_{\beta\gamma} \qquad \partial_1 : U_{\alpha\beta\gamma} \hookrightarrow U_{\alpha\gamma} \qquad \partial_2 : U_{\alpha\beta\gamma} \hookrightarrow U_{\alpha\beta}$$

Under the contravariant functor Ω^*, these inclusions induce maps

$$\Omega^*(M) \xrightarrow{r} \prod \Omega^*(U_{\alpha_0}) \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} \prod_{\alpha_0<\alpha_1} \Omega^*(U_{\alpha_0\alpha_1}) \overset{\delta_0}{\underset{\delta_2}{\overset{\delta_1}{\Rrightarrow}}} \prod_{\alpha_0<\alpha_1<\alpha_2} \Omega^*(U_{\alpha_0\alpha_1\alpha_2}) \Rrightarrow \cdots$$

where δ_i are the corresponding restriction maps, eg.

$$\delta_0 : \Omega^*(U_{\beta\gamma}) \to \prod_{\alpha<\beta} \Omega^*(U_{\alpha\beta\gamma})$$
$$\delta_1 : \Omega^*(U_{\alpha\gamma}) \to \prod_{\alpha<\beta<\gamma} \Omega^*(U_{\alpha\beta\gamma})$$
$$\delta_2 : \Omega^*(U_{\alpha\beta}) \to \prod_{\beta<\gamma} \Omega^*(U_{\alpha\beta\gamma})$$

Analogously to the Mayer-Vietoris sequence, we define the difference operator

$$\delta = \sum_i (-1)^i \delta_i : \prod \Omega^*(U_{\alpha_0\ldots\alpha_p}) \to \prod \Omega^*(U_{\alpha_0\ldots\alpha_{p+1}}).$$

Explicitly, if $\omega \in \prod \Omega^*(U_{\alpha_0\ldots\alpha_p})$ has components $\omega_{\alpha_0\ldots\alpha_p}$, then

$$(\delta\omega)_{\alpha_0\ldots\alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_{p+1}}, \tag{$*$}$$

where terms on the right are restricted to $U_{\alpha_0\ldots\alpha_{p+1}}$, and $\widehat{\cdot}$ denotes omission. We check that $\delta^2 = 0$:

$$\begin{aligned}(\delta^2\omega)_{\alpha_0\ldots\alpha_{p+2}} &= \sum_i (-1)^i (\delta\omega)_{\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_{p+2}} \\ &= \sum_{j<i} (-1)^{i+j} \omega_{\alpha_0\ldots\widehat{\alpha_j}\ldots\widehat{\alpha_i}\ldots\alpha_{p+2}} + \sum_{j>i} (-1)^{i+j-1} \omega_{\alpha_0\ldots\widehat{\alpha_i}\ldots\widehat{\alpha_j}\ldots\alpha_{p+2}} = 0.\end{aligned}$$

Here we set a convention: if $\omega \in \prod \Omega^*(U_{\alpha_0\ldots\alpha_p})$ and π is a permutation of $\{0, \ldots, p\}$, define

$$\omega_{\alpha_{\pi(0)}\ldots\alpha_{\pi(p)}} = (-1)^{\sigma(\pi)} \omega_{\alpha_0\ldots\alpha_p},$$

where $\sigma(\pi)$ is the signature of π. It can be checked that $(*)$ still holds when indices are interpreted with this convention.

Proposition (Generalised Mayer-Vietoris sequence). *The sequence*

$$0 \longrightarrow \Omega^*(M) \xrightarrow{r} \prod \Omega^*(U_{\alpha_0}) \xrightarrow{\delta} \prod \Omega^*(U_{\alpha_0\alpha_1}) \xrightarrow{\delta} \prod \Omega^*(U_{\alpha_0\alpha_1\alpha_2}) \xrightarrow{\delta} \cdots$$

is exact.

Proof. Note that elements of $\prod \Omega^*(U_{\alpha_0})$ annihilated by δ are those which agree on all overlaps $U_{\alpha_0\alpha_1}$, which are precisely those which can be glued together to give a global form on M. Hence $\operatorname{im} r = \ker \delta \cap \prod \Omega^*(U_{\alpha_0})$.

Let $\{\rho_\alpha\}$ be a partition of unity subordinate to the open cover $\mathfrak{U} = \{U_\alpha\}$. Define an operator $K : \prod \Omega^*(U_{\alpha_0\ldots\alpha_p}) \to \prod \Omega^*(U_{\alpha_0\ldots\alpha_{p-1}})$ by

$$(K\omega)_{\alpha_0\ldots\alpha_{p-1}} = \sum_\alpha \rho_\alpha \omega_{\alpha\alpha_0\ldots\alpha_{p-1}}.$$

Then

$$\begin{aligned}
(\delta K\omega)_{\alpha_0\ldots\alpha_p} &= \sum_i (-1)^i (K\omega)_{\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_p} \\
&= \sum_{i,\alpha} (-1)^i \rho_\alpha \omega_{\alpha\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_{p-1}}, \\
(K\delta\omega)_{\alpha_0\ldots\alpha_p} &= \sum_\alpha \rho_\alpha (\delta\omega)_{\alpha\alpha_0\ldots\alpha_{p-1}} \\
&= \left(\sum_\alpha \rho\alpha\right) \omega_{\alpha_0\ldots\alpha_p} + \sum_{i,\alpha} (-1)^{i+1} \rho_\alpha \omega_{\alpha\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_p} \\
&= \omega_{\alpha_0\ldots\alpha_p} - \sum_{i,\alpha} (-1)^i \rho_\alpha \omega_{\alpha\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_p}.
\end{aligned}$$

Hence $\delta K + K\delta = 1$. In particular, if $\delta\omega = 0$ then $\delta(K\omega) = \omega$, so every cocycle is a coboundary, and the given sequence is exact. □

We can arrange the Mayer-Vietoris sequence in an augmented double complex:

$$\begin{array}{ccc|ccc}
 & & & q & & \\
 & \vdots & & \vdots & \vdots & \iddots \\
0 \to & \Omega^2(M) & \xrightarrow{r} & K^{0,2} & K^{1,2} & \cdots \\
0 \to & \Omega^1(M) & \xrightarrow{r} & K^{0,1} & K^{1,1} & \cdots \\
0 \to & \Omega^0(M) & \xrightarrow{r} & K^{0,0} & K^{1,0} & \cdots \\
\hline
 & & & & & p
\end{array}$$

where $K^{p,q} = \prod \Omega^q(U_{\alpha_0\ldots\alpha_p})$ are the p-cochains of the open cover $\mathfrak{U}$ with values in the q-forms. As before, there are two differential operators associated to this complex: the exterior derivative d along the columns, and the difference operator δ along the rows.

Note that d and δ commute; hence we can define a (singly graded) differential complex $K^n = \bigoplus_{p+q=n} K^{p,q}$, with differential operator $D = \delta + (-1)^p d$ satisfying

$$D^2 = (\delta + (-1)^{p+1} d)\delta + (-1)^p(\delta + (-1)^p d)d = \delta^2 \pm d\delta \mp \delta d + d^2 = 0.$$

The double complex $K^{*,*}$ is called the *Čech-de Rham* complex. The exactness of the rows implies that the Čech-de Rham complex computes the de Rham cohomology of M; more precisely, we have the following:

Proposition (Generalised Mayer-Vietoris principle). *The map of cohomologies*

$$r^* : H^*_{dR}(M) \to H_D(K^*),$$

induced by the inclusion map $r : \Omega^*(M) \to K^{*,*}$, *is an isomorphism.*

Proof. Note that $Dr = (\delta + d)r = dr = rd$. Thus r is a chain map, so r^* is well-defined.

Consider $\varphi = \sum_{p=0}^{n} \varphi_p \in \ker D$, with $\varphi_p \in K^{p,n-p}$. If p' is the maximal index with $\varphi_{p'} \neq 0$, and $p' \geqslant 1$, we have

$$0 = D\varphi = \delta\varphi_{p'} + (\delta\varphi_{p'-1} + (-1)^{p'} d\varphi_{p'}) + (\delta\varphi_{p'-2} + (-1)^{p'-1} d\varphi_{p'-1}) + \cdots,$$

where bracketed terms have the same order (ie. belong to the same $K^{p,q}$). Hence $\delta\varphi_{p'} = 0$, so by exactness of rows there exists $\psi \in K^{p'-1,n-p'}$ with $\delta\psi = \varphi_{p'}$:

$$\begin{array}{|lllll} \varphi_0 & & & & \\ & \ddots & & & \\ & & \varphi_{p'-1} & & \\ & & \psi \quad \varphi_{p'} & & \\ & & & 0 & \\ & & & & \ddots \\ \hline \end{array} \qquad \begin{array}{|lllll} \varphi_0 & & & & \\ & \ddots & & & \\ & & * & & \\ & & & 0 & \\ & & & & 0 \\ & & & & \quad \ddots \\ \hline \end{array}$$

Hence $\varphi - D\psi$ is an element of the D-cohomology class of φ with $K^{p,n-p}$ components 0 for $p \geqslant p'$. Repeating this argument, we see that every D-cohomology class has a representative φ whose only nonzero component is the top component φ_0. In particular, this shows r^* is surjective.

$$0 \to \omega \xrightarrow{r} \begin{array}{|llll} r(\omega) & & & \\ \; * & 0 & & \\ & * & 0 & \\ & & \ddots & \ddots \\ \hline \end{array} \qquad \begin{array}{r} 0 \to \omega \xrightarrow{r} \\ 0 \to \tau \xrightarrow{r} \\ \\ \\ \end{array} \begin{array}{|llll} r(\omega) & & & \\ \; \varphi & 0 & & \\ & 0 & 0 & \\ & & \ddots & \ddots \\ \hline \end{array}$$

Moreover, if $r^*(\omega) = 0$, ie. $r(\omega) = D\varphi$ for some φ, then by changing representatives in the D-cohomology class, we may assume that $\varphi \in K^{0,n}$. Then taking the $K^{1,n}$ component of the above gives $\delta\varphi = 0$, ie. $\varphi = r(\tau)$ for some τ. Hence $\omega = d\tau$, so ω is cohomologous to 0. Thus r^* is injective. □

Čech cohomology

We now augment the Čech-de Rham complex with the kernel of the bottom d in each column:

$$
\begin{array}{ccccccc}
 & q & & & & & \\
\vdots & \Big| & \vdots & \vdots & \vdots & \iddots & \\
0 \to \Omega^2(M) \xrightarrow{r} & \Big| & \prod \Omega^2(U_{\alpha_0}) & \prod \Omega^2(U_{\alpha_0\alpha_1}) & \prod \Omega^2(U_{\alpha_0\alpha_1\alpha_2}) & \cdots & \\
0 \to \Omega^1(M) \xrightarrow{r} & \Big| & \prod \Omega^1(U_{\alpha_0}) & \prod \Omega^1(U_{\alpha_0\alpha_1}) & \prod \Omega^1(U_{\alpha_0\alpha_1\alpha_2}) & \cdots & \\
0 \to \Omega^0(M) \xrightarrow{r} & \Big| & \prod \Omega^0(U_{\alpha_0}) & \prod \Omega^0(U_{\alpha_0\alpha_1}) & \prod \Omega^0(U_{\alpha_0\alpha_1\alpha_2}) & \cdots & p \\
 & & i \uparrow & i \uparrow & i \uparrow & & \\
 & & C^0(\mathfrak{U}, \mathbb{R}) & C^1(\mathfrak{U}, \mathbb{R}) & C^2(\mathfrak{U}, \mathbb{R}) & & \\
 & & \uparrow & \uparrow & \uparrow & & \\
 & & 0 & 0 & 0 & &
\end{array}
$$

Note that $C^p(\mathfrak{U}, \mathbb{R})$ is the vector space of functions which are locally constant on each $U_{\alpha_0 \dots \alpha_p}$. The bottom row

$$C^0(\mathfrak{U}, \mathbb{R}) \xrightarrow{\delta} C^1(\mathfrak{U}, \mathbb{R}) \xrightarrow{\delta} C^2(\mathfrak{U}, \mathbb{R}) \xrightarrow{\delta} \cdots$$

is a differential complex, and its cohomology $H^*(\mathfrak{U}, \mathbb{R})$ is called the *Čech cohomology* of the open cover $\mathfrak{U}$.

Assume that $\mathfrak{U}$ is a *good cover*, ie. all finite intersections $U_{\alpha_0 \dots \alpha_p}$ are diffeomorphic to $\mathbb{R}^n$. Then the augmented columns of the double complex are all exact, and the same argument as in the previous section will give an isomorphism between the cohomology of the double complex and the Čech cohomology, ie.

$$H^*_{dR}(M) \cong H_D(K^*) \cong H^*(\mathfrak{U}, \mathbb{R}).$$

The importance of this isomorphism comes from the link between de Rham cohomology, which describes the differential geometry of forms on M, and Čech cohomology, which is determined by purely combinatorial data, namely how open sets in $\mathfrak{U}$ intersect each other.

Examples

To highlight the combinatorial nature of the Čech cohomology, we will compute a few examples explicitly.

Example: $H^*(S^1)$ Consider $S^1 = \mathbb{R}/\mathbb{Z}$, with the open cover $\mathfrak{U} = \{U_0, U_1, U_2\}$ given by $U_0 = (-1/3, 1/3)$, $U_1 = (0, 2/3)$, $U_2 = (1/3, 1)$. It is easy to check that $\mathfrak{U}$ is a good cover, with $U_{\alpha\beta} \neq \emptyset$, $U_{012} = \emptyset$. Now

$$C^0(\mathfrak{U}, \mathbb{R}) = \{(\omega_0, \omega_1, \omega_2) : \omega_\alpha \text{ constant on } U_\alpha\} \cong \mathbb{R}^3,$$
$$C^1(\mathfrak{U}, \mathbb{R}) = \{(\eta_{01}, \eta_{02}, \eta_{12}) : \eta_{\alpha\beta} \text{ constant on } U_{\alpha\beta}\} \cong \mathbb{R}^3.$$

Now the coboundary operator $\delta : C^0 \to C^1$ is given by $(\delta\omega)_{\alpha\beta} = \omega_\beta - \omega_\alpha$, so

$$\ker \delta = \{(c, c, c) : c \in \mathbb{R}\} \cong \mathbb{R},$$
$$\operatorname{im} \delta = \frac{C^0(\mathfrak{U}, \mathbb{R})}{\ker(\delta : C^0 \to C^1)} \cong \mathbb{R}^2.$$

Hence

$$H^0(S^1) = H^0(\mathfrak{U}, \mathbb{R}) = \ker(\delta : C^0 \to C^1) \cong \mathbb{R},$$
$$H^1(S^1) = H^1(\mathfrak{U}, \mathbb{R}) = \frac{\ker(\delta : C^1 \to C^2)}{\operatorname{im}(\delta : C^0 \to C^1)} \cong \frac{\mathbb{R}^3}{\mathbb{R}^2} \cong \mathbb{R}.$$

Example: $H^*(S^2)$ Consider S^2 as the surface of a sphere embedded in $\mathbb{R}^3$. Inscribe a regular tetrahedron, and project it outwards onto S^2. Take open sets $\mathfrak{U} = \{U_0, U_1, U_2, U_3\}$ slightly bigger than the four projected faces; then $\mathfrak{U}$ is a good cover of S^2. As above, we have $U_{\alpha\beta\gamma} \neq \emptyset$, $U_{0123} = \emptyset$. Hence

$$C^0(\mathfrak{U}, \mathbb{R}) \cong \mathbb{R}^4, \qquad C^1(\mathfrak{U}, \mathbb{R}) \cong \mathbb{R}^6, \qquad C^2(\mathfrak{U}, \mathbb{R}) \cong \mathbb{R}^4.$$

Then

$$\ker(\delta : C^0 \to C^1) = \{(c, c, c, c) \, : \, c \in \mathbb{R}\} \cong \mathbb{R},$$
$$\operatorname{im}(\delta : C^0 \to C^1) = \frac{C^0(\mathfrak{U}, \mathbb{R})}{\ker(\delta : C^0 \to C^1)} \cong \mathbb{R}^3.$$

If $\eta = (\eta_{01}, \eta_{02}, \eta_{03}, \eta_{12}, \eta_{13}, \eta_{23}) \in \ker(\delta : C^1 \to C^2)$, then we have

$$\eta_{01} - \eta_{02} + \eta_{12} = 0 \qquad\qquad \eta_{01} - \eta_{03} + \eta_{13} = 0$$
$$\eta_{02} - \eta_{03} + \eta_{23} = 0 \qquad\qquad \eta_{12} - \eta_{13} + \eta_{23} = 0$$

Hence

$$\ker(\delta : C^1 \to C^2) = \{(a, b, c, b - a, c - a, c - b) \, : \, a, b, c \in \mathbb{R}\} \cong \mathbb{R}^3,$$
$$\operatorname{im}(\delta : C^1 \to C^2) = \frac{C^1(\mathfrak{U}, \mathbb{R})}{\ker(\delta : C^1 \to C^2)} \cong \mathbb{R}^3.$$

Combining the above, we get

$$H^0(S^2) = H^0(\mathfrak{U}, \mathbb{R}) = \ker(\delta : C^0 \to C^1) \cong \mathbb{R},$$
$$H^1(S^2) = H^1(\mathfrak{U}, \mathbb{R}) = \frac{\ker(\delta : C^1 \to C^2)}{\operatorname{im}(\delta : C^0 \to C^1)} \cong \frac{\mathbb{R}^3}{\mathbb{R}^3} \cong 0,$$
$$H^2(S^2) = H^2(\mathfrak{U}, \mathbb{R}) = \frac{\ker(\delta : C^2 \to C^3)}{\operatorname{im}(\delta : C^1 \to C^2)} \cong \frac{\mathbb{R}^4}{\mathbb{R}^3} \cong \mathbb{R}.$$

References

R. Bott, L. W. Tu, *Differential Forms in Algebraic Topology*, Springer-Verlag, 1982.