# Differential Geometry and Electromagnetism

MA5216 Presentation

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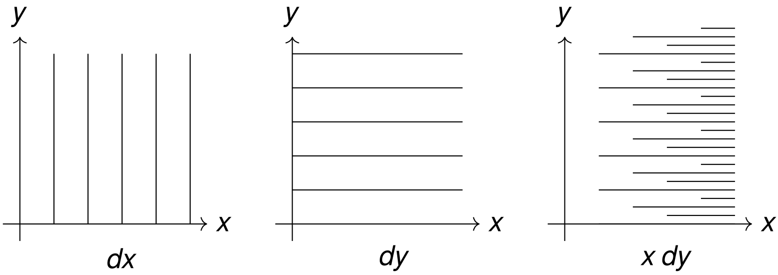
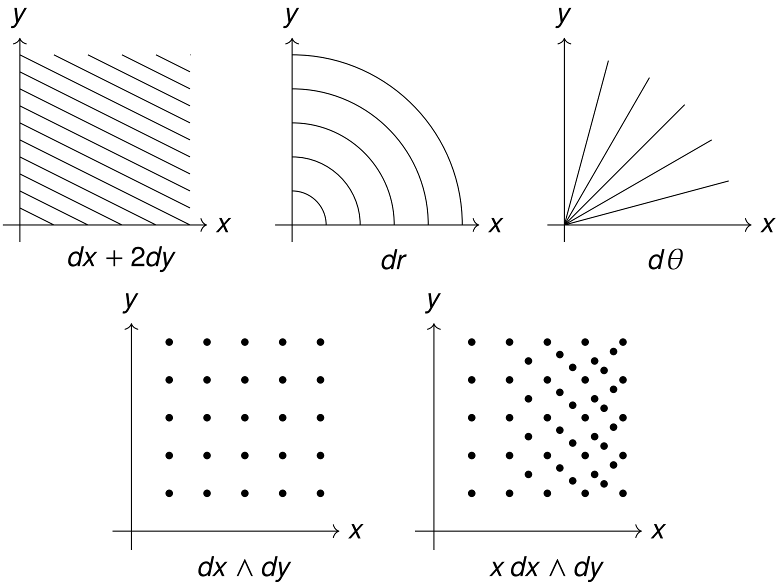
Apr 2018

# Visualising differential forms

- *k*-form: something that can be integrated over oriented *k*-submanifolds
- Geometrical picture (Piponi 1998)




$k$-forms $\leftrightarrow$ codim-$k$ oriented submanifolds


y
x
$dx + 2dy$
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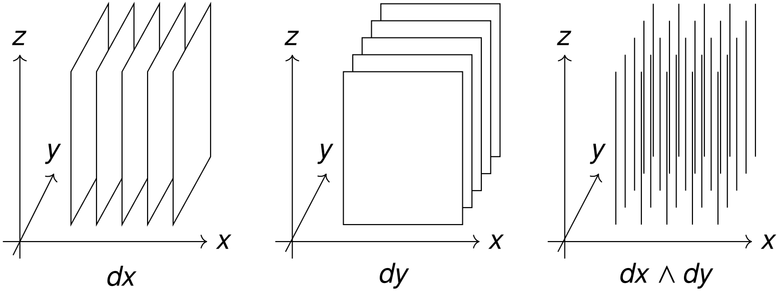
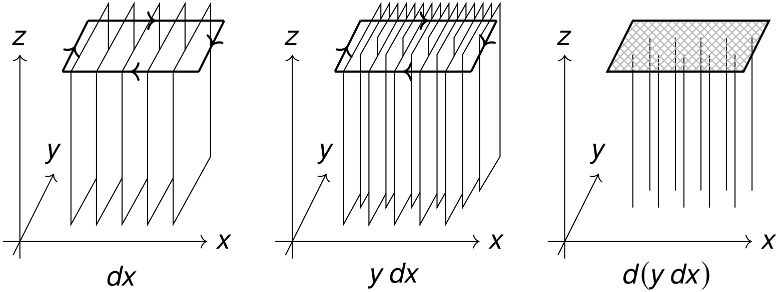
x
$dx \wedge dy$
y
x
$x\, dx \wedge dy$




Wedge product ↔ intersection




Exterior derivative $\leftrightarrow$ boundary operator

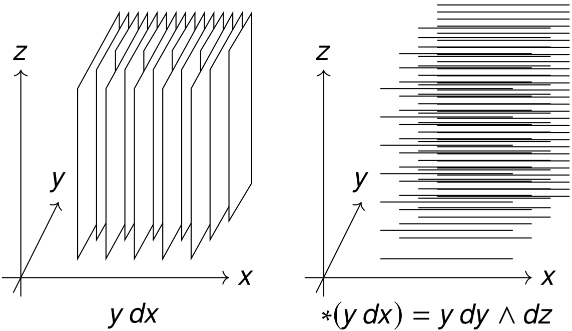
$d^2 = 0 \leftrightarrow \partial^2 = 0$

# Hodge duality

$\alpha$ $k$-form $\leftrightarrow *\alpha$ $(n-k)$-form,

$$\alpha \wedge (*\beta) = \langle \alpha, \beta \rangle \omega$$

($\omega$ $n$-form, $\langle \alpha_1 \wedge \cdots \wedge \alpha_k, \beta_1 \wedge \cdots \wedge \beta_k \rangle = \det(\langle \alpha_i, \beta_j \rangle)$)



$$\operatorname{grad} f = \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$$

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{pmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{pmatrix}$$

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\oint_{\partial S} \mathbf{F} \cdot d\ell = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

$$\oiint_{\partial V} \mathbf{F} \cdot d\mathbf{S} = \iiint_V (\nabla \cdot \mathbf{F})\, dV$$

$$\Omega_0 \xrightarrow{\text{grad}} \Omega_1 \xrightarrow{\text{curl}} \Omega_2 \xrightarrow{\text{div}} \Omega_3$$

$$\oint_{\partial V} \mathcal{F} = \int_V d\mathcal{F}$$

# Maxwell's equations

$$\oiint_{\partial V} \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \iiint_V \rho \, dV \qquad \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\oiint_{\partial V} \mathbf{B} \cdot d\mathbf{S} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

$$\oint_{\partial S} \mathbf{E} \cdot d\ell = -\frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{S} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\oint_{\partial S} \mathbf{B} \cdot d\ell = \mu_0 \left( \iint_S \mathbf{J} \cdot d\mathbf{S} + \varepsilon_0 \frac{d}{dt} \iint_S \mathbf{E} \cdot d\mathbf{S} \right) \qquad \nabla \times \mathbf{B} = \mu_0 \left( \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

**E**, **B** each represent two different physical quantities!
(field = 1-form, flux = 2-form)

| | | | |
|---|---|---|---|
| Electric field intensity | $\mathbf{E}$ | $\mathcal{E}$ | 1-form |
| Magnetic field intensity | $\mathbf{H}$ | $\mathcal{H}$ | 1-form |
| Electric flux intensity | $\mathbf{D}$ | $\mathcal{D}$ | 2-form |
| Magnetic flux intensity | $\mathbf{B}$ | $\mathcal{B}$ | 2-form |
| Electric current density | $\mathbf{J}$ | $\mathcal{J}$ | 2-form |
| Electric charge density | $\rho$ | $\rho$ | 3-form |

Constitutive relations:
$$\mathcal{D} = \varepsilon_0 * \mathcal{E}$$
$$\mathcal{B} = \mu_0 * \mathcal{H}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad d\mathcal{D} = \rho$$
$$\nabla \cdot \mathbf{B} = 0 \qquad d\mathcal{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad d\mathcal{E} = -\frac{\partial \mathcal{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \mu_0 \left( \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \qquad d\mathcal{H} = \mathcal{J} + \frac{\partial \mathcal{D}}{\partial t}$$

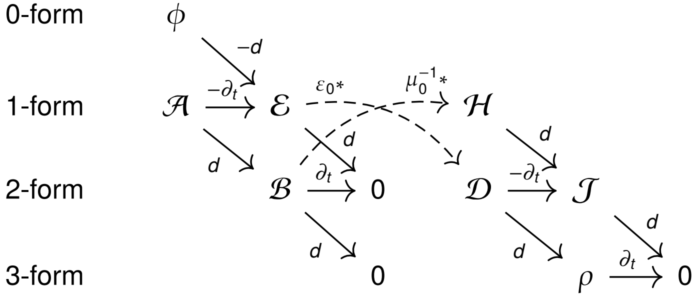
Poincaré lemma:

$$d\mathcal{D} = \rho$$

$$d\mathcal{B} = 0 \qquad \mathcal{B} = d\mathcal{A} \qquad \text{Magnetic potential } \mathcal{A} \quad \text{1-form}$$

$$d\mathcal{E} = -\frac{\partial \mathcal{B}}{\partial t} \qquad \mathcal{E} = -d\phi - \frac{\partial \mathcal{A}}{\partial t} \qquad \text{Electric potential } \phi \quad \text{0-form}$$

$$d\mathcal{H} = \mathcal{J} + \frac{\partial \mathcal{D}}{\partial t}$$



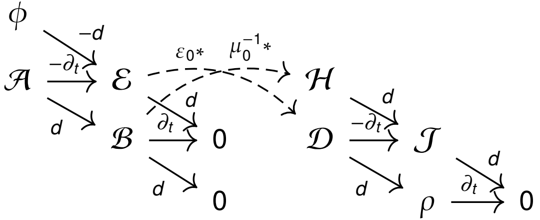
Consider a form $\omega$ in $\mathbb{R}^3$ as a form in $\mathbb{R}^4$:

$$d_4\omega = d_3\omega + dt\,\frac{\partial \omega}{\partial t}$$

$$d_4(dt\,\omega) = -dt\,d_3\omega$$

$$\alpha = \mathcal{A} - dt\,\phi \qquad \Psi = \mathcal{D} + dt\,\mathcal{H}$$

$$\Phi = \mathcal{B} - dt\,\mathcal{E} \qquad \gamma = \rho - dt\,\mathcal{J}$$




$$\alpha \xrightarrow{d} \Phi \xrightarrow{d} 0 \qquad \Psi \xrightarrow{d} \gamma \xrightarrow{d} 0$$

# Problems with classical electromagnetism

- Lorentz force: $m\frac{\partial \mathbf{v}}{\partial t} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$
- Speed of light: $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$
- Singularities in the electromagnetic field

# Special relativity

Postulates:

- The laws of physics are the same for two observers moving at constant velocity relative to each other.
- The speed of light is a constant, independent of the state of motion of the light source.

Coordinates: $\mathbf{x} = (ct, x, y, z) = (x^0, x^1, x^2, x^3)$
Coordinate transform between inertial frames: $\mathbf{x}' = A\mathbf{x}$

# Lorentz transformations

Speed of light: $(\eta = \mathrm{diag}(1, -1, -1, -1))$

$$x'^2 + y'^2 + z'^2 = c^2 t'^2 \iff x^2 + y^2 + z^2 = c^2 t^2$$

$$\mathbf{x} A \eta A^T \mathbf{x}^T = 0 \iff \mathbf{x} \eta \mathbf{x}^T = 0$$

$$A \eta A^T = a \eta$$

Take two moving reference frames:

$$a(v_{12}) = \frac{a(v_1)}{a(v_2)} \implies a \text{ constant} \implies a \equiv 1$$

$\langle \mathbf{x}, \tilde{\mathbf{x}} \rangle = \mathbf{x} \eta \tilde{\mathbf{x}}^T = g_{ij} x^i \tilde{x}^j$ covariant metric
$(ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$ covariant);

$$A \in \mathrm{SO}(1,3) = \left\langle \mathrm{SO}(3), \begin{pmatrix} \gamma & \gamma v/c & & \\ \gamma v/c & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\rangle \quad (\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}})$$

Consider a particle moving on the path

$$\Gamma(\theta) = (ct(\theta), x(\theta), y(\theta), z(\theta))$$

relative to rest frame $(ct, x, y, z)$.
Fix point on $\Gamma$, take comoving frame $(ct', x', y', z')$:

$$c^2 dt'^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

$$\implies dt' = \sqrt{1 - \frac{v^2}{c^2}}\, dt = \frac{dt}{\gamma}$$

Hence proper time $\tau(\theta) := \int_{t_0}^{t} \frac{dt}{\gamma}$ is invariant.
Note that

$$d\tau = \frac{ds}{c} = \frac{dt}{\gamma}, \qquad \frac{dt}{d\tau} = \gamma, \qquad \frac{dx}{d\tau} = \gamma \frac{dx}{dt}$$

Vectors which transform like the position vector:

- 4-position $(ct, x, y, z) = (x^0, x^1, x^2, x^3)$
- 4-velocity $u_i = \frac{dx_i}{d\tau}$
  - $u = (\gamma c, \gamma \frac{dx}{dt}, \gamma \frac{dy}{dt}, \gamma \frac{dz}{dt})$
  - $\langle u, u \rangle = c^2$
- 4-momentum $P_i = mu_i$
- 4-acceleration $\frac{du_i}{d\tau}$
- 4-force $\frac{dP_i}{d\tau}$
- ...

- Principle of least action: physical path taken is stationary point of action functional
$$S = \int_a^b L(x^i, \dot{x}^i, \theta)\, d\theta,$$
where dot stands for $\frac{\partial}{\partial\theta}$
- Euler-Lagrange equations:
$$\frac{d}{d\theta}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0$$
- If $S$ is invariant, solution trajectory is covariant

# Example 1

$$S = \int -mc\, ds$$
$$= \int -mc\sqrt{v^i v_i}\, d\theta \qquad (v_i = \frac{dx^i}{d\theta})$$

Euler-Lagrange:

$$\frac{d}{d\theta}\frac{\partial L}{\partial v_i} = \frac{\partial L}{\partial x_i}$$
$$\frac{d}{d\theta} mc \frac{v^i}{\sqrt{v^i v_i}} = 0$$

Parametrise such that $\theta \to \tau$ on trajectory:

$$v^i \to u^i, \quad \sqrt{v^i v_i} \to c$$
$$mu^i = \text{const}$$

# Example 2

Let $\underline{A} := A_j dx^j$ be a 1-form:

$$S = \int \left( -mc\, ds - \frac{q}{c} A_j\, dx^j \right)$$
$$= -\int \left( mc\sqrt{v^i v_i} + \frac{q}{c} A_j v^j \right) d\theta \qquad (v_i = \frac{dx^i}{d\theta})$$

Euler-Lagrange:

$$\frac{d}{d\theta}\frac{\partial L}{\partial v_i} = \frac{\partial L}{\partial x_i}$$
$$\frac{d}{d\theta}\left( mc\frac{v^i}{\sqrt{v^i v_i}} + \frac{q}{c} A^i \right) = \frac{q}{c}\frac{\partial A^j}{\partial x_i} v_j$$

Parametrise such that $\theta \rightarrow \tau$ on trajectory:

$$m\frac{du^i}{d\tau} = \frac{q}{c}\left( \frac{\partial A^j}{\partial x_i} - \frac{\partial A^i}{\partial x_j} \right) u_j$$

$$m\frac{du^i}{d\tau} = \frac{q}{c}F^{ij}u_j, \qquad F^{ij} = \frac{\partial A^j}{\partial x_i} - \frac{\partial A^i}{\partial x_j}$$

Write $u = (\gamma c, \gamma \frac{dx}{dt}, \gamma \frac{dy}{dt}, \gamma \frac{dz}{dt}) = (u_0, \vec{u}) = \gamma(c, \vec{v})$:

$$\begin{aligned} m\frac{d\vec{u}}{d\tau} &= \frac{q}{c}\left(-u_0\begin{pmatrix}F^{01}\\F^{02}\\F^{03}\end{pmatrix} - \begin{pmatrix}F^{21}u_2 - F^{13}u_3\\F^{32}u_3 - F^{21}u_1\\F^{13}u_1 - F^{32}u_2\end{pmatrix}\right) \\ &= q\left(\frac{u_0}{c}\begin{pmatrix}F_{01}\\F_{02}\\F_{03}\end{pmatrix} + \vec{u}\times\begin{pmatrix}F_{32}/c\\F_{13}/c\\F_{21}/c\end{pmatrix}\right) \\ &= \gamma q(\mathbf{E} + \vec{v}\times\mathbf{B}). \end{aligned}$$

$$(F_{ij}) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -cB_z & cB_y \\ -E_y & cB_z & 0 & -cB_x \\ -E_z & -cB_y & cB_x & 0 \end{pmatrix} \quad \text{electromagnetic tensor}$$

$$\mathbf{E} = (F_{01}, F_{02}, F_{03}) \qquad c\mathbf{B} = (F_{32}, F_{13}, F_{21})$$

$$\mathcal{E} = F_{01} dx^1 + \cdots \qquad -c\mathcal{B} = F_{23} dx^2 dx^3 + \cdots$$

$$\begin{aligned} \underline{F} := d_4 \underline{A} &= d_4(A_i dx^i) \\ &= \sum_{i<j} \left( \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} \right) dx^i \, dx^j \\ &= \sum_{i<j} F_{ij} dx^i \, dx^j = dx^0 \mathcal{E} - c\mathcal{B} \end{aligned}$$

Hence we can take $\underline{A} = dx^0 \phi - c\mathcal{A}$.

$$*_3 dx^1 = dx^2 dx^3 \qquad \underline{*} dx^1 = dx^0 dx^2 dx^3 \qquad \underline{*} dx^0 dx^1 = -dx^2 dx^3$$
$$*_3 dx^1 dx^2 = dx^3 \qquad \underline{*} dx^1 dx^2 = dx^0 dx^3 \qquad \underline{*} dx^0 dx^1 dx^2 = dx^3$$

$$\begin{aligned} \underline{*F} &= \underline{*}(dx^0 \mathcal{E} - c\mathcal{B}) \\ &= -c\, dx^0 *_3 \mathcal{B} - *_3 \mathcal{E} \\ &= -\sqrt{\frac{\mu_0}{\varepsilon_0}}(dx^0\, \mathcal{H} + c\mathcal{D}) \end{aligned}$$

$$\underline{A} = dx^0 \phi - c\mathcal{A} \qquad \underline{G} = -dx^0\, \mathcal{H} - c\mathcal{D}$$
$$\underline{F} = dx^0 \mathcal{E} - c\mathcal{B} \qquad \underline{J} = dx^0 \mathcal{J} - c\rho$$

$$\underline{A} \xrightarrow{d} \underline{F} \xrightarrow{d} 0 \qquad \underline{F} \xrightarrow{\sqrt{\mu_0/\varepsilon_0}\,\underline{*}} \underline{G} \qquad \underline{G} \xrightarrow{d} \underline{J} \xrightarrow{d} 0$$

4-potential $A^i = (\phi, c\mathbf{A})$, 4-current $(-\underline{*}\underline{J})^i = (c\rho, \mathbf{J})$

# Maxwell's equations in relativity

$$d_4\underline{F} = 0$$

$$d_4\underline{*F} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\underline{J}$$

# References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