Matrix Lie Groups

MA5210 Presentation

Ang Yan Sheng
A0144836Y

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Matrix Lie groups

A *matrix Lie group* is a closed subgroup of $\mathrm{GL}(n,\mathbb{R})$ or $\mathrm{GL}(n,\mathbb{C})$ (under vector norm induced from $\mathbb{R}^{n^2}$ or $\mathbb{C}^{n^2}$).

Example

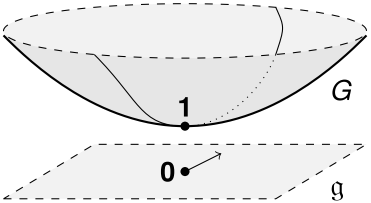
$$\mathrm{GL}(n,\mathbb{R}) \qquad \mathrm{GL}(n,\mathbb{C})$$

$$\mathrm{SL}(n,\mathbb{R}) = \{X : \det(X) = 1\} \qquad \mathrm{SL}(n,\mathbb{C}) = \{X : \det(X) = 1\}$$

$$\mathrm{O}(n) = \{X : XX^T = \mathbf{1}\} \qquad \mathrm{U}(n) = \{X : XX^\dagger = \mathbf{1}\}$$

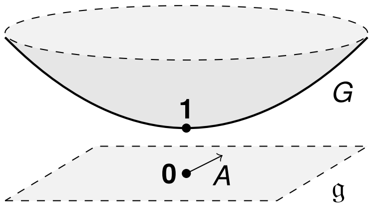
$$\mathrm{SO}(n) = \{X : XX^T = \mathbf{1}, \det(X) = 1\} \qquad \mathrm{SU}(n) = \{X : XX^\dagger = \mathbf{1}, \det(X) = 1\}$$

$\mathrm{SO}(p,q), \mathrm{SU}(p,q), \mathrm{Sp}(n), \ldots$



$$\mathfrak{g} = T_{\mathbf{1}}G = \left\{ C'(0) \;:\; \begin{matrix} C : (-\delta, \delta) \to G \text{ smooth} \\ C(0) = \mathbf{1} \end{matrix} \right\}.$$

The matrix exponential

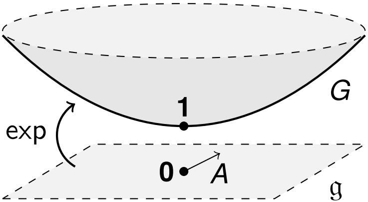


$$\mathbf{1} + \frac{A}{n} \quad \text{is “almost” in } G.$$

$$\left(\mathbf{1} + \frac{A}{n}\right)^n \text{ is “almost” in } G.$$

“ $\exp(A)$ is in G.”

The matrix exponential



Definition

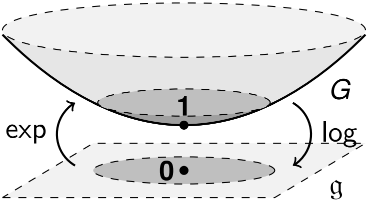
$$\exp(A) = \mathbf{1} + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots .$$

Proposition

$$A \in \mathfrak{g} \implies \exp(A) \in G.$$

$$\exp(A + B) = \exp(A)\exp(B) \quad \text{if} \quad AB = BA.$$

The matrix logarithm



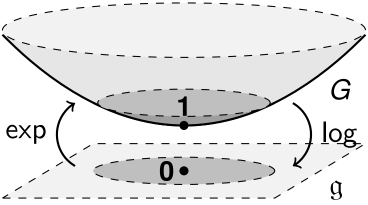
Definition

$$\log(\mathbf{1} + X) = X - \frac{X^2}{2} + \frac{X^3}{3} - + \cdots \qquad (|X| < 1).$$

Proposition

For some $\varepsilon > 0$, $\log(N_\varepsilon(\mathbf{1})) \subseteq \mathfrak{g}$.

The matrix logarithm



Definition

$$\log(\mathbf{1} + X) = X - \frac{X^2}{2} + \frac{X^3}{3} - + \cdots \qquad (|X| < 1).$$

Theorem

G is a smooth manifold, with $\dim G = \dim \mathfrak{g}$*.*

Hence every matrix Lie group is a Lie group.

The Lie bracket

Can we recover G from vector space $\mathfrak{g}$? No.
Need new operation on $\mathfrak{g}$:

- Non-commutative
- "Captures" group law on G

Proposition

The commutator $[X, Y] = XY - YX$ is a bilinear map on $\mathfrak{g}$, with

$$[X, Y] = -[Y, X],$$
$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

Hence every $\mathfrak{g}$ is a Lie algebra.

The Lie bracket gives the group law near **1**:

Theorem (Campbell-Baker-Hausdorff)

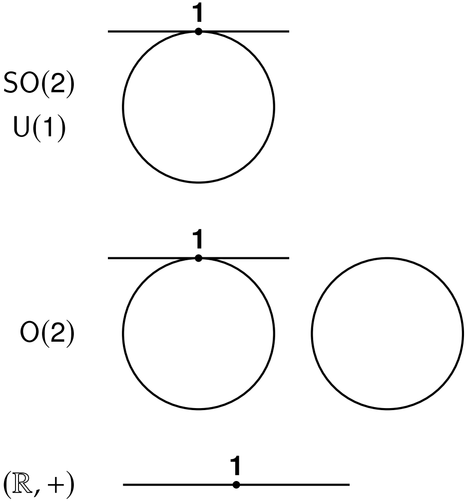
If $\exp(Z) = \exp(X)\exp(Y)$, *then*

$$Z = X + Y + \frac{[X,Y]}{2} + \frac{[X,[X,Y]] + [Y,[Y,X]]}{12} + \cdots,$$

where all terms are Lie brackets in X and Y.

Can we recover G from Lie algebra $\mathfrak{g}$? No!

Examples: 1D



- $\left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right\} \cong (\mathbb{R}, +)$
- O(2) is not connected
- SO(2) $\cong \mathbb{R}/\mathbb{Z}$: quotient by discrete central subgroup
- $\mathbb{R}$ is simply connected, SO(2) is not

Quarternions

Complex numbers $\mathbb{C}$

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a\mathbf{1} + b\mathbf{i}$$

Quarternions $\mathbb{H}$

$$\begin{pmatrix} a + id & -b - ic \\ b - ic & a - id \end{pmatrix} = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1},$$

$$\mathbf{ij} = \mathbf{k}, \quad \mathbf{jk} = \mathbf{i}, \quad \mathbf{ki} = \mathbf{j},$$

$$\mathbf{ji} = -\mathbf{k}, \quad \mathbf{kj} = -\mathbf{i}, \quad \mathbf{ik} = -\mathbf{j}.$$

$$|q| = \sqrt{a^2 + b^2 + c^2 + d^2}$$

$$\overline{q} = a\mathbf{1} - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$$

$$q\overline{q} = |q|^2\mathbf{1}$$

SU(2) and SO(3)

$$\begin{aligned}\mathrm{SU}(2) &= \{q \in \mathbb{H} \;:\; |q| = 1\} \\ &= \{\cos\theta + u\sin\theta \;:\; \theta \in \mathbb{R},\, u \in \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k},\, |u| = 1\}.\end{aligned}$$

Theorem

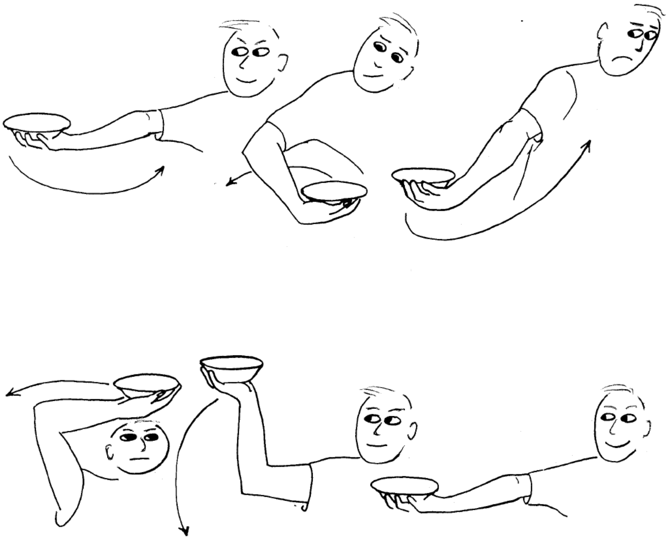
Let $t = \cos\theta + u\sin\theta \in \mathrm{SU}(2)$. *Then on* $\mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$,

$$\rho_t : q \mapsto t^{-1}qt$$

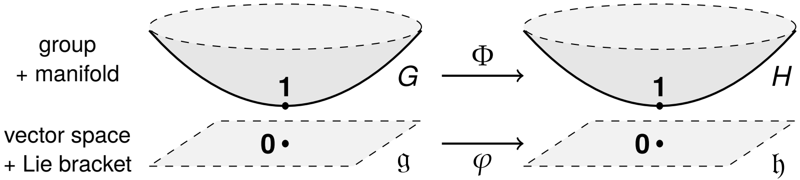
is a rotation of angle 2θ *about the axis* u.
Moreover, $\rho_t = \rho_{t'} \iff t' = \pm t$.

- $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\{\pm\mathbf{1}\}$: quotient by discrete central subgroup
- $\mathrm{SU}(2) \approx \mathbb{S}^3$ is simply connected
- $\mathrm{SO}(3) \approx \mathbb{S}^3/\{\pm 1\} = \mathbb{RP}^3$ is not

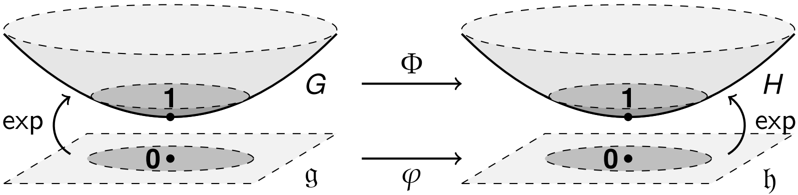
The plate trick



Lie homomorphisms



The main theorem

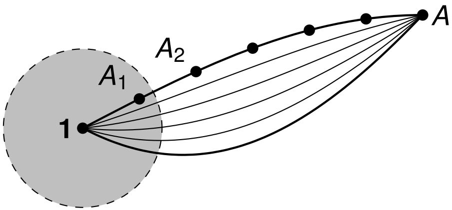


Theorem

Let G, H be simply connected matrix Lie groups. Then every Lie homomorphism $\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$ determines a Lie homomorphism $\Phi : G \longrightarrow H$ that induces φ.

Step 1: Lift φ to Φ near **1**.

By Campbell-Baker-Hausdorff, group law is preserved.



Step 2: Define $\Phi(A)$ by stepping along a path.

- $\Phi(A)$ is invariant under refinements.
- $\Phi(A)$ does not depend on choice of steps.
- $\Phi(A)$ is invariant under small deformation of paths.
- $\Phi(A)$ does not depend on choice of path.

Corollary

Any simply connected matrix Lie group is determined by its Lie algebra.

Conclusion

- Matrix Lie groups: large class of examples
- Constructions for general Lie groups: $\exp$, Lie bracket, $\ldots$
- Geometry $\leftrightarrow$ algebra $\leftrightarrow$ topology

References

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