Matrix Lie Groups

MA5210 Reading Report 1

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This report will be a brief introduction to the central ideas of Lie theory, by developing them in the concrete context of *matrix groups* (ie. subgroups of $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$).

Some classical groups

The most natural and important examples of matrix groups are those that preserve certain properties of the space they are acting on. We start by listing some of these examples, called the *classical groups* by Hermann Weyl, who gave the first systematic treatment of these groups in the 1930s.

For instance, over the vector space $\mathbb{R}^n$, we have the *general linear* and *special linear groups*:

$$\begin{aligned} \mathrm{GL}(n, \mathbb{R}) &= \{X \in M_n(\mathbb{R}) \,:\, \det(X) \neq 0\}, \\ \mathrm{SL}(n, \mathbb{R}) &= \{X \in M_n(\mathbb{R}) \,:\, \det(X) = 1\}. \end{aligned}$$

The elements of $\mathrm{GL}(n, \mathbb{R})$ are the invertible linear maps (which preserve the vector space structure of $\mathbb{R}^n$), and the elements of $\mathrm{SL}(n, \mathbb{R})$ additionally preserve volume and orientation.

$\mathbb{R}^n$ also has the structure of an inner product space, so it is possible to talk about length and angles. The symmetries of this space form the *orthogonal* and *special orthogonal groups*:

$$\begin{aligned} \mathrm{O}(n) &= \{X \in M_n(\mathbb{R}) \,:\, XX^T = \mathbf{1}\}, \\ \mathrm{SO}(n) &= \{X \in M_n(\mathbb{R}) \,:\, XX^T = \mathbf{1},\, \det(X) = 1\}. \end{aligned}$$

By standard results in linear algebra, the elements of $\mathrm{O}(n)$ are the isometries of $\mathbb{R}^n$, and the elements of $\mathrm{SO}(n)$ additionally preserve orientation.

There are analogous constructions for all 4 families of groups above for $\mathbb{C}^n$, with the Hermitian form replacing the role of the bilinear form:

$$\begin{aligned} \mathrm{GL}(n, \mathbb{C}) &= \{X \in M_n(\mathbb{C}) \,:\, \det(X) \neq 0\}, \\ \mathrm{SL}(n, \mathbb{C}) &= \{X \in M_n(\mathbb{C}) \,:\, \det(X) = 1\}, \\ \mathrm{U}(n) &= \{X \in M_n(\mathbb{C}) \,:\, XX^\dagger = \mathbf{1}\}, \\ \mathrm{SU}(n) &= \{X \in M_n(\mathbb{C}) \,:\, XX^\dagger = \mathbf{1},\, \det(X) = 1\}, \end{aligned}$$

where $X^\dagger$ denotes the conjugate transpose of X.

The term "classical group" usually also includes the symmetry groups of indefinite forms (such as $\mathrm{SO}(p,q)$ and $\mathrm{SU}(p,q)$), and also analogues over the quarternions $\mathbb{H}^n$ (such as $\mathrm{Sp}(n)$). These groups can also be studied with the methods in this report (for example, they are also matrix Lie groups, to be defined later); however, in the interest of space, they lie beyond the scope of this report.

The matrix exponential

We would like to define the exponential on $M_n(\mathbb{C})$ by a power series, which means we need a metric on $M_n(\mathbb{C})$ to deal with convergence.

For a matrix $A = (a_{ij}) \in M_n(\mathbb{C})$, define its *absolute value* as

$$|A| = \sqrt{\mathrm{Tr}(AA^\dagger)} = \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{1/2}.$$

Note that this is the Euclidean norm under the usual isomorphism $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$. This gives $M_n(\mathbb{C})$ the structure of a normed vector space, for which we recall the following results from functional analysis (proofs omitted):

Proposition. *Let* $A, X_k \in M_n(\mathbb{C})$*. Then:*

(a) Absolute convergence implies convergence, ie.

$$\textstyle\sum_{k=0}^{\infty} |X_k| < \infty \implies \sum_{k=0}^{\infty} X_k \text{ converges.}$$

(b) If $\sum_{k=0}^{\infty} X_k$ *converges, then*

$$\textstyle\sum_{k=0}^{\infty} AX_k = A\sum_{k=0}^{\infty} X_k, \qquad \sum_{k=0}^{\infty} X_k A = \left(\sum_{k=0}^{\infty} X_k\right) A.$$

The key property of the matrix absolute value is submultiplicativity:

Proposition. *For any* $A, B \in M_n(\mathbb{C})$*, we have* $|AB| \leqslant |A||B|$.

Proof. Write $A = (a_{ij})$, $B = (b_{ij})$, and $AB = (c_{ij})$. Then

$$\begin{aligned} |c_{ij}|^2 = \left|\sum_{k=1}^{n} a_{ik}b_{kj}\right|^2 &\leqslant \left(\sum_{k=1}^{n} |a_{ik}||b_{kj}|\right)^2 \\ &\leqslant \left(\sum_{k=1}^{n} |a_{ik}|^2\right)\left(\sum_{k=1}^{n} |b_{kj}|^2\right) \quad \text{(Cauchy-Schwarz).} \end{aligned}$$

Hence

$$\begin{aligned} |AB|^2 = \sum_{i,j=1}^{n} |c_{ij}|^2 &\leqslant \sum_{i,j=1}^{n} \left(\sum_{k=1}^{n} |a_{ik}|^2\right)\left(\sum_{k=1}^{n} |b_{kj}|^2\right) \\ &= \left(\sum_{i,k=1}^{n} |a_{ik}|^2\right)\left(\sum_{k,j=1}^{n} |b_{kj}|^2\right) = |A|^2|B|^2. \end{aligned}$$

$\square$

We may now define the *matrix exponential function* for $A \in M_n(\mathbb{C})$ by

$$\exp(A) = I_n + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots .$$

Proposition. *The series for* $\exp(A)$ *converges for all* $A \in M_n(\mathbb{C})$.

Proof. It suffices to check that the series is absolutely convergent. Indeed, we have

$$\begin{aligned}|I_n| + |A| + \left|\frac{A^2}{2!}\right| + \left|\frac{A^3}{3!}\right| + \cdots &\leqslant \sqrt{n} + |A| + \frac{|A|^2}{2!} + \frac{|A|^3}{3!} + \cdots \\ &= \sqrt{n} - 1 + e^{|A|} < \infty.\end{aligned}$$

□

The following are some useful properties of the matrix exponential.

Proposition. *Let* $A, B \in M_n(\mathbb{C})$. *Then:*

(a) *If* $AB = BA$, *then* $\exp(A)\exp(B) = \exp(A + B)$.
(b) $\exp(A)$ *is invertible, with inverse* $\exp(-A)$.
(c) $\frac{d}{dt}\exp(tA) = A\exp(tA) = \exp(tA)A$.
(d) *If* A *is invertible, then* $\exp(ABA^{-1}) = A\exp(B)A^{-1}$.
(e) $\det(\exp(A)) = e^{\mathrm{Tr}(A)}$.

Proof. (a) Note that the identity $\exp(x)\exp(y) = \exp(x + y)$ holds over $\mathbb{C}$, so it holds for formal power series in two commuting variables x, y. The result follows.

(b) This follows from (a), since A and $-A$ commute.

(c) Note that each entry of the matrix $\exp(tA)$ is a power series in t, and thus can be differentiated term-by-term in its disk of convergence (ie. for all $t \in \mathbb{C}$). Hence

$$\begin{aligned}\frac{d}{dt}\exp(tA) &= \frac{d}{dt}\left(I_n + tA + t^2\frac{A^2}{2!} + t^3\frac{A^3}{3!} + \cdots\right) \\ &= A + tA^2 + t^2\frac{A^3}{2!} + \cdots \\ &= A\exp(tA) = \exp(tA)A,\end{aligned}$$

since A can be factored out either on the left or the right.

(d) We have

$$\begin{aligned}\exp(ABA^{-1}) &= I_n + ABA^{-1} + \frac{(ABA^{-1})^2}{2!} + \frac{(ABA^{-1})^3}{3!} + \cdots \\ &= I_n + ABA^{-1} + A\frac{B^2}{2!}A^{-1} + A\frac{B^3}{3!}A^{-1} + \cdots = A\exp(B)A^{-1}.\end{aligned}$$

(e) By the Jordan normal form, we can write $A = PUP^{-1}$ for some invertible P, where $U = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. Now

$$\exp(U) = \sum_{k=0}^{\infty} \frac{U^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} \lambda_1^k & & * \\ & \ddots & \\ 0 & & \lambda_n^k \end{pmatrix} = \begin{pmatrix} e^{\lambda_1} & & * \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix},$$

so we have

$$\begin{aligned} \det(\exp(A)) &= \det(\exp(PUP^{-1})) \\ &= \det(P \exp(U) P^{-1}) \\ &= \det(\exp(U)) \\ &= e^{\lambda_1} \cdots e^{\lambda_n} = e^{\lambda_1 + \cdots + \lambda_n} = e^{\mathrm{Tr}(A)}. \end{aligned}$$ □

The Lie algebra of a matrix group

Tangent spaces For any matrix group G, we define its *tangent space at the identity* $T_\mathbf{1}G$ as the set of matrices of the form $A'(0)$ for some smooth path $A : (-\varepsilon, \varepsilon) \to G$ with $A(0) = \mathbf{1}$. Since matrix groups are not necessarily manifolds, we first need to show that $T_\mathbf{1}G$ is indeed a vector space.

Proposition. $T_\mathbf{1}G$ *is a* $\mathbb{R}$*-vector space.*

Proof. Let $X, Y \in T_\mathbf{1}G$, so there exists smooth paths A, B in G with $A(0) = B(0) = \mathbf{1}$, $A'(0) = X$, $B'(0) = Y$. Hence $C(t) = A(t)B(t)$ is also a smooth path in G, with $C(0) = \mathbf{1}$ and

$$C'(0) = A'(0)B(0) + A(0)B'(0) = X + Y,$$

so $X + Y \in T_\mathbf{1}G$.

Also, if $r \in \mathbb{R}$, then $D(t) = A(rt)$ is a smooth path in G with $D(0) = \mathbf{1}$ and $D'(0) = rA'(0) = rX$, so $rX \in T_\mathbf{1}G$. □

Lie algebras Knowing $T_\mathbf{1}G$ gives us some information about G. However, $T_\mathbf{1}G$ cannot capture the full behaviour of G, since addition in $T_\mathbf{1}G$ commutes while multiplication in G might not. To capture this nonabelian behaviour, we define the *commutator* of two matrices X, Y by $[X, Y] = XY - YX$.

It is straightforward to check that the commutator is a bilinear map on $M_n(\mathbb{C})$, satisfying the following properties:

Proposition. *Let* $X, Y, Z \in M_n(\mathbb{C})$*. Then*

$$[X, Y] = -[Y, X], \qquad [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

A vector space equipped with a bilinear form $[\cdot, \cdot]$ satisfying the above two conditions is known as a *Lie algebra*.

Proposition. T_1G *is closed under the commutator, ie. if* $X, Y \in T_1G$ *then* $[X, Y] \in T_1G$.

Proof. Let A, B be smooth paths in G such that $A(0) = B(0) = \mathbf{1}$, $A'(0) = X$ and $B'(0) = Y$. For fixed s, consider the smooth path $C_s(t) = A(s)B(t)A(s)^{-1}$ in G, which satisfies $C(0) = \mathbf{1}$. Hence

$$D(s) = C_s'(0) = A(s)YA(s)^{-1} \in T_1G,$$

so D is a smooth path in the vector space T_1G. Hence

$$D'(0) = A'(0)YA(0)^{-1} + A(0)Y(-A'(0)) = XY - YX = [X, Y] \in T_1G. \qquad \square$$

Hence the vector space T_1G, equipped with the commutator, is a Lie algebra called the *Lie algebra of* G. It is denoted by the corresponding Fraktur letter in lowercase, such as $\mathfrak{g}$.

The Lie algebras of classical groups

We now compute the Lie algebras for the classical groups that we have seen so far. The main idea is to use the differential equation for the matrix exponential, to construct a smooth path with a given velocity at $\mathbf{1}$.

$\mathbf{GL}(n, \mathbb{R})$, $\mathbf{GL}(n, \mathbb{C})$ When $G = GL(n, \mathbb{R})$, we expect the tangent space to be the whole of $M_n(\mathbb{R})$. Indeed, for any $X \in M_n(\mathbb{R})$, consider the smooth path $A(t) = \exp(tX)$. Then $A(t)$ is an $n \times n$ invertible matrix, so A is a smooth path in $GL(n, \mathbb{C})$. Also, $A(0) = \mathbf{1}$ and

$$A'(0) = \frac{d}{dt} \exp(tX)\big|_{t=0} = X \exp(tX)\big|_{t=0} = X,$$

so $X \in \mathfrak{gl}(n, \mathbb{R})$). An analogous argument holds for $GL(n, \mathbb{C})$, so

$$\boxed{\begin{aligned} \mathfrak{gl}(n, \mathbb{R}) &= M_n(\mathbb{R}), \\ \mathfrak{gl}(n, \mathbb{C}) &= M_n(\mathbb{C}). \end{aligned}}$$

The differential of det Before proceeding, we will use the same method to find the differential of the determinant map at $\mathbf{1}$.

Proposition. *For* $X \in M_n(\mathbb{C})$*, we have* $(T\det)_{\mathbf{1}}(X) = \mathrm{Tr}(X)$.

Proof. Let $A(t) = \exp(tX)$ as before. Then $A(0) = \mathbf{1}$ and $A'(0) = X$, so

$$\begin{aligned} (T\det)_{\mathbf{1}}(X) &= \frac{d}{dt} \det(A(t))\big|_{t=0} \\ &= \frac{d}{dt} \det(\exp(tX))\big|_{t=0} \\ &= \frac{d}{dt} e^{t\,\mathrm{Tr}(X)}\Big|_{t=0} = \mathrm{Tr}(X). \end{aligned} \qquad \square$$

$\mathbf{SL}(n,\mathbb{R})$, $\mathbf{SL}(n,\mathbb{C})$ Let $C(t)$ be a smooth path in $SL(n,\mathbb{R})$. Then $\det(C(t)) = 1$, and applying $d/dt|_{t=0}$ on both sides gives

$$\frac{d}{dt}\det(C(t))\big|_{t=0} = \mathrm{Tr}(C'(0)) = 0,$$

so any $X \in \mathfrak{sl}(n,\mathbb{R})$ satisfies $\mathrm{Tr}(X) = 0$.

Conversely, for any $X \in M_n(\mathbb{R})$ with $\mathrm{Tr}(X) = 0$, let $A(t) = \exp(tX)$. Then

$$\det(A(t)) = \det(\exp(tX)) = e^{t\,\mathrm{Tr}(X)} = 1,$$

so $A(t)$ is a smooth path in $SL(n,\mathbb{R})$. Also, $A(0) = \mathbf{1}$ and $A'(0) = X$, so $X \in \mathfrak{sl}(n,\mathbb{R})$. We argue analogously for $SL(n,\mathbb{C})$ to obtain

$$\boxed{\begin{aligned}\mathfrak{sl}(n,\mathbb{R}) &= \{X \in M_n(\mathbb{R}) \,:\, \mathrm{Tr}(X) = 0\},\\ \mathfrak{sl}(n,\mathbb{C}) &= \{X \in M_n(\mathbb{C}) \,:\, \mathrm{Tr}(X) = 0\}.\end{aligned}}$$

$\mathbf{O}(n)$, $\mathbf{SO}(n)$ Let $C(t)$ be a smooth path in $O(n)$ or $SO(n)$. Then $C(t)C(t)^T = \mathbf{1}$, and applying $d/dt|_{t=0}$ on both sides gives

$$\frac{d}{dt}C(t)C(t)^T\big|_{t=0} = C'(0)C(0)^T + C(0)C'(0)^T = C'(0) + C'(0)^T = \mathbf{0},$$

so any $X \in \mathfrak{o}(n)$ or $\mathfrak{so}(n)$ satisfies $X + X^T = \mathbf{0}$, ie. X is *skew-symmetric*.

Conversely, for any $X \in M_n(\mathbb{R})$ with $X + X^T = \mathbf{0}$, let $A(t) = \exp(tX)$. Then

$$A(t)A(t)^T = \exp(tX)\exp(tX^T) = \exp(tX)\exp(-tX) = \exp(\mathbf{0}) = \mathbf{1},$$

so $A(t)$ is a smooth path in $O(n)$. Hence $\det(A(t)) = \pm 1$, but since A is continuous and $\det(A(0)) = 1$, we have $\det(A(t)) = 1$ for all t, so $A(t)$ is also a smooth path in $SO(n)$. Also, $A(0) = \mathbf{1}$ and $A'(0) = X$, so $X \in \mathfrak{o}(n)$ and $\mathfrak{so}(n)$. Hence

$$\boxed{\mathfrak{o}(n) = \mathfrak{so}(n) = \{X \in M_n(\mathbb{C}) \,:\, X + X^T = \mathbf{0}\}.}$$

$\mathbf{U}(n)$, $\mathbf{SU}(n)$ By replacing the transpose X^T with the conjugate transpose $X^\dagger$, we may argue completely analogously to the above to obtain

$$\boxed{\begin{aligned}\mathfrak{u}(n) &= \{X \in M_n(\mathbb{C}) \,:\, X + X^\dagger = \mathbf{0}\},\\ \mathfrak{su}(n) &= \{X \in M_n(\mathbb{C}) \,:\, X + X^\dagger = \mathbf{0},\ \mathrm{Tr}(X) = 0\}.\end{aligned}}$$

The matrix logarithm

Matrix Lie groups We would like to say that the Lie algebra contains all the information about a given matrix group. This is not true in general (eg. $O(n)$ and $SO(n)$ have the same Lie algebra). However, it is a general phenomenon of Lie theory that given certain topological conditions on the group, the structure of the group is completely determined by its Lie algebra.

In the case of matrix groups, it is necessary to assume a certain closure property. We say that a matrix group G is a *matrix Lie group* if G is a closed subset of $GL(n,\mathbb{R})$

or $\mathrm{GL}(n, \mathbb{C})$ under the matrix absolute value. By usual results in metric spaces, we have the following:

Proposition. *A matrix group G is a matrix Lie group if and only if it is* closed under nonsingular limits, *ie. if $A_1, A_2, \ldots \in G$ satisfies $\lim_{n\to\infty} A_n = A$ and $\det(A) \neq 0$, then $A \in G$.*

Corollary. *All the classical groups $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$ are matrix Lie groups.*

Proof. As an illustrative example, we will prove that $\mathrm{SO}(n)$ is a matrix Lie group; an analogous argument can be applied to the other cases.

If $A_1, A_2, \ldots \in \mathrm{SO}(n)$ is a sequence converging to $A \in \mathrm{GL}(n, \mathbb{R})$, then

$$\begin{cases} A_m A_m^T = \mathbf{1} \\ \det(A_m) = 1 \end{cases} \implies \begin{cases} AA^T = \mathbf{1} \\ \det(A) = 1, \end{cases}$$

by continuity of $X \mapsto XX^T$ and $X \mapsto \det(X)$, respectively. Hence $A \in \mathrm{SO}(n)$, so $\mathrm{SO}(n)$ is a matrix Lie group by the previous proposition. □

The matrix logarithm For $A \in M_n(\mathbb{C})$ with $|A| < 1$, we define the *matrix logarithm function* by

$$\log(\mathbf{1} + A) = A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + - \cdots .$$

This series is absolutely convergent, by comparison with the geometric series $|A| + |A|^2 + |A|^3 + \cdots$. Hence $\log(\mathbf{1} + A)$ is well-defined and continuous on $|A| < 1$.

Proposition. *Let $X, Y \in M_n(\mathbb{C})$. Then:*

(a) If $\log(X)$ is defined, then $\exp(\log(X)) = X$.
(b) If $\log(\exp(X))$ is defined, then $\log(\exp(X)) = X$.
(c) If $XY = YX$, and $\log(X)$, $\log(Y)$ and $\log(XY)$ are all defined, then $\log(XY) = \log(X) + \log(Y)$.

Proof. Note that these identities hold in formal power series in two commuting variables, eg. in $\mathbb{C}$. □

Theorem. *If G is a matrix Lie group and $X \in T_{\mathbf{1}}G$, then $\exp(X) \in G$.*

Proof. Let A be a smooth path in G with $A(0) = \mathbf{1}$ and $A'(0) = X$. Note that

$$\frac{\log(A(1/n))}{1/n} = \frac{A(1/n) - \mathbf{1}}{1/n} \left(\mathbf{1} - \frac{A(1/n) - \mathbf{1}}{2} + \frac{(A(1/n) - \mathbf{1})^2}{3} - + \cdots \right).$$

Now as $n \to \infty$, we have $A(1/n) \to \mathbf{1}$. Note that the norms of the terms in the brackets (other than the first) is bounded by a geometric sequence, so taking limits on both sides gives

$$\lim_{n\to\infty} n \log(A(1/n)) = \lim_{n\to\infty} \frac{A(1/n) - \mathbf{1}}{1/n} = A'(0) = X.$$

Taking exp on both sides yields

$$\lim_{n\to\infty} \exp(n \log(A(1/n))) = \lim_{n\to\infty} A(1/n)^n = \exp(X).$$

Now $A(1/n) \in G$ implies $A(1/n)^n \in G$, and since $\exp(X)$ is invertible (with inverse $\exp(-X)$), we have $\exp(X) \in G$ by closure under nonsingular limits. $\square$

Proposition. *Let* G *be a matrix Lie group, and consider the sequences* $A_m \in G$ *and* $\alpha_m \in \mathbb{R}$ *such that*

$$\lim_{m\to\infty} A_m = \mathbf{1} \quad \text{and} \quad \lim_{m\to\infty} \frac{A_m - \mathbf{1}}{\alpha_m} = X.$$

Then $\exp(tX) \in G$ *for all* $t \in \mathbb{R}$. *In particular,* $X \in T_{\mathbf{1}}G$.

Proof. Let $a_m = \lfloor 1/\alpha_m \rfloor$, so

$$\lim_{m\to\infty} a_m(A_m - \mathbf{1}) = X + \lim_{m\to\infty} \left(a_m - \frac{1}{\alpha_m}\right)(A_m - \mathbf{1}) = X,$$

since $|a_m - 1/\alpha_m| < 1$ and $A_m - \mathbf{1} \to \mathbf{0}$.

As before, note that

$$\frac{\log(A_m)}{1/a_m} = \frac{A_m - \mathbf{1}}{1/a_m}\left(\mathbf{1} - \frac{A_m - \mathbf{1}}{2} + \frac{(A_m - \mathbf{1})^2}{3} - + \cdots\right),$$

and the sum in the bracket has limit $\mathbf{1}$ as $m \to \infty$. Hence

$$\lim_{m\to\infty} a_m \log(A_m) = \lim_{m\to\infty} a_m(A_m - \mathbf{1}) = X,$$

and taking exp on both sides gives

$$\exp(X) = \lim_{m\to\infty} A_m^{a_m}.$$

Now $A_m \in G$ implies $A_m^{a_m} \in G$, and since $\exp(X)$ is invertible, we have $\exp(X) \in G$ by closure under nonsingular limits.

Replacing α_m with α_m/t in the above argument, we also get $\exp(tX) \in G$. Hence the smooth path $A(t) = \exp(tX)$ lies in G, and satisfies $A(0) = \mathbf{1}$, $A'(0) = X$. Hence $X \in T_{\mathbf{1}}G$. $\square$

Proposition. *Let* G *be a matrix group. Then there is a neighbourhood* $U \subseteq G$ *of* $\mathbf{1}$ *such that* $\log(U) \subseteq T_{\mathbf{1}}G$.

Proof. Assume for sake of contradiction that there is no such neighbourhood of $\mathbf{1}$. Then there is a sequence $A_m \in G$ such that $\lim_{m\to\infty} A_m = \mathbf{1}$ and $\log(A_m) \notin T_{\mathbf{1}}G$.

Write $\log(A_m) = X_m + Y_m$, with $X_m \in T_{\mathbf{1}}G$ and $Y_m \in T_{\mathbf{1}}G^{\perp}$, the orthogonal complement of $T_{\mathbf{1}}G$ in $M_n(\mathbb{C})$. Hence $Y_m \neq \mathbf{0}$. Since log is continuous, we have $A_m \to \mathbf{1}$ implies $X_m, Y_m \to \mathbf{0}$ as $m \to \infty$.

Note that for every m, the matrix $Z_m = Y_m/|Y_m|$ has norm 1. Hence by Bolzano-Weierstrass, the sequence Z_m has a convergent subsequence Z_{m_k}. Replacing the sequences X_m, Y_m by X_{m_k}, Y_{m_k}, we may define

$$\lim_{m\to\infty} \frac{Y_m}{|Y_m|} = Y,$$

so that Y lies in $T_1G^\perp$ and has norm 1.

Now let $T_m = \exp(-X_m)A_m$, which is in G since $X_m \in T_1G$. Also,

$$\begin{aligned} T &= \exp(-X_m)\exp(X_m + Y_m) \\ &= \left(\mathbf{1} - X_m + \frac{X_m^2}{2} + \cdots\right)\left(1 + X_m + Y_m + \frac{(X_m+Y_m)^2}{2} + \cdots\right) \\ &= \mathbf{1} + Y_m + (*). \end{aligned}$$

Note that in the above expansion, the powers of X_m that appear are those which appear in the power series expansion of

$$\exp(-X_m)\exp(X_m) = \mathbf{1},$$

so every term in $(*)$ contains either a Y_m^2 or a X_mY_m. Hence

$$\lim_{m\to\infty} \frac{T_m - \mathbf{1}}{|Y_m|} = \lim_{m\to\infty} \frac{Y_m}{|Y_m|} = Y,$$

but by the previous proposition the limit on the left belongs to T_1G, contradiction.

Hence there is a neighbourhood of $\mathbf{1}$ in G which gets sent into T_1G by log. □

Theorem. *Let G be a matrix Lie group. Then $\log$ is a homeomorphism from a neighbourhood of $\mathbf{1}$ in G and a neighbourhood of $\mathbf{0}$ in T_1G.*

Proof. Take $U \subseteq G$ as in the previous proposition. Then $\log|_U$ is a continuous map from U to $\log(U) \subseteq T_1G$ with inverse exp, so $\log(U)$ is a neighbourhood of $\mathbf{0}$, and $\log|_U : U \to \log(U)$ is a homeomorphism. □

Theorem. *Every matrix Lie group G is a smooth manifold, with $\dim G = \dim \mathfrak{g}$.*

Proof. Firstly, note that $G \subseteq GL(n, \mathbb{C})$, so G is second-countable and Hausdorff.

Take $U \subseteq G$ as in the previous proposition. For each $A \in G$, let $U_A = AU = \{AX : X \in U\}$, and $\varphi_A : U_A \to M_n(\mathbb{C})$ by

$$\varphi_A(AX) = \log(X).$$

Note that this is a homeomorphism from U_A to a neighbourhood of $\mathbf{0}$ in $\mathfrak{g}$.

Now for any $A, B \in G$ with $U_A \cap U_B \neq \emptyset$, and any $X \in \varphi_B(U_A \cap U_B)$, we have

$$(\varphi_A \circ \varphi_B^{-1})(X) = \log(A^{-1}B\exp(X)),$$

which is a smooth function in its domain of definition (since exp, log and left multiplication by $A^{-1}B$ are smooth functions in their respective domains). Hence $\{(U_A, \varphi_A) : A \in G\}$ is a smooth atlas for G, so G is a smooth manifold with dimension $\dim \mathfrak{g}$. □

Lie groups A *Lie group* is a smooth manifold G with a group structure such that the group multiplication $\mu : G \times G \to G$, $\mu(a, b) = ab$, and group inverse $\iota : G \to G$, $\iota(a) = a^{-1}$, are smooth. It is not hard to see that matrix Lie groups are Lie groups, because the matrix multiplication and inverse operations are smooth on $GL(n, \mathbb{C})$.

Lie homomorphisms

Let G and H be Lie groups. A *Lie group homomorphism* is a group homomorphism $\Phi : G \to H$ which is also a smooth map between manifolds. In the context of matrix Lie groups, the smoothness requirement can be rephrased as follows (by chain rule): the map $A'(0) \mapsto (\Phi \circ A)'(0)$ is a well-defined linear map from $\mathfrak{g}$ to $\mathfrak{h}$.

Let $\mathfrak{a}$ and $\mathfrak{b}$ be Lie algebras. A *Lie algebra homomorphism* is an algebra homomorphism $\varphi : \mathfrak{a} \to \mathfrak{b}$ which preserves the Lie bracket.

We are now able to state the main theorems of this report. Firstly, isomorphic Lie groups have isomorphic Lie algebras. However, we have already noted that the converse is false, since matrix Lie groups are not determined by their Lie algebra. Nevertheless, we will prove that *simply-connected* Lie groups with isomorphic Lie algebras are indeed isomorphic.

For convenience, we will refer to both Lie group homomorphisms and Lie algebra homomorphisms as Lie homomorphisms; the intended meaning will always be clear from context. As usual, in this section $\mathfrak{g}$ always denotes the Lie algebra of the matrix Lie group G (since we have yet to define the Lie algebra of a general Lie group).

Proposition. *Let* G, H *be matrix Lie groups. Then any Lie homomorphism* $\Phi : G \to H$ *induces a Lie homomorphism* $\varphi : \mathfrak{g} \to \mathfrak{h}$*, in the sense that*

$$\varphi(A'(0)) = (\Phi \circ A)'(0)$$

for any smooth path A *in* G *through* $\mathbf{1}$*.*

Proof. Since $\Phi : G \to H$ is smooth, φ is well-defined. Hence it remains to show that φ preserves the Lie bracket.

If A, B are smooth paths in G passing through $\mathbf{1}$, let

$$C_s(t) = A(s)B(t)A(s)^{-1} \qquad \text{for fixed } s.$$

Now $C_s'(0) = A(s)B'(0)A(s)^{-1} \in \mathfrak{g}$, so

$$\begin{aligned} \varphi(C_s'(0)) &= \frac{d}{dt}\, \Phi(A(s))\Phi(B(t))\Phi(A(s))^{-1}\Big|_{t=0} \\ &= \Phi(A(s))(\Phi \circ B)'(0)\Phi(A(s))^{-1} \in \mathfrak{h}. \end{aligned}$$

Let $D(s) = C_s'(0)$, so $D(s)$ is a smooth path in $\mathfrak{g}$ and $\varphi(D(s))$ is a smooth path in $\mathfrak{h}$. By linearity of φ,

$$\varphi(D'(0)) = \frac{d}{ds}\, \varphi(D(s))\big|_{s=0},$$

where $D'(0) = A'(0)B'(0) - B'(0)A'(0) = [A'(0), B'(0)]$ and

$$\begin{aligned} \frac{d}{ds}\, \varphi(D(s))\big|_{s=0} &= (\Phi \circ A)'(0)(\Phi \circ B)'(0) - (\Phi \circ B)'(0)(\Phi \circ A)'(0) \\ &= [(\Phi \circ A)'(0), (\Phi \circ B)'(0)] \\ &= [\varphi(A'(0)), \varphi(B'(0))], \end{aligned}$$

and we are done. □

Corollary. *If* G, H *are isomorphic Lie groups, then* $\mathfrak{g}, \mathfrak{h}$ *are isomorphic Lie algebras.*

Proof. Let $\Phi : G \to H$ be a Lie isomorphism, so Φ, Φ^{-1} induces homomorphisms $\varphi : \mathfrak{g} \to \mathfrak{h}$, $\psi : \mathfrak{h} \to \mathfrak{g}$ respectively. Now

$$\psi((\Phi \circ A)'(0)) = (\Phi^{-1} \circ \Phi \circ A)'(0) = A'(0),$$

so $\psi = \varphi^{-1}$, and thus φ is a Lie isomorphism from $\mathfrak{g}$ to $\mathfrak{h}$. □

To prove the partial converse of the above statement, we need a hard and rather surprising theorem that expresses the product $\exp(X)\exp(Y)$ as a explicit series, which only involves Lie brackets in X and Y. We omit the proof of this theorem; an elementary approach, due to Eichler, is presented in Stillwell (2008).

Theorem (Campbell-Baker-Hausdorff). *Given* $X, Y, Z \in M_n(\mathbb{C})$ *such that* $\exp(Z) = \exp(X)\exp(Y)$. *Then*

$$Z = X + Y + \frac{1}{2}[X, Y] + (*),$$

where the higher-order terms in $(*)$ *are all linear combinations of Lie brackets in* X *and* Y.

We can now show our main result. Recall that a topological space is *simply connected* if it is path-connected, and any two paths with the same endpoints can be continuously deformed into each other (ie. are in the same homotopy class).

Theorem. *Let* G, H *are simply connected matrix Lie groups. Then every Lie homomorphism* $\varphi : \mathfrak{g} \to \mathfrak{h}$ *determines a Lie homomorphism* $\Phi : G \to H$ *that induces* φ.

Proof (sketch). Let $U = B(\mathbf{1}, \delta) \subseteq G$ be a neighbourhood of $\mathbf{1}$ such that $\log|_U$ is a local homeomorphism.

1. Let $\Phi(\exp(X)) = \exp(\varphi(X))$ for all $\exp(X) \in U$. Then

$$\begin{aligned}\Phi(\exp(X)\exp(Y)) &= \exp(\varphi(X + Y + [X, Y]/2 + \cdots)) \\ &= \exp(\varphi(X) + \varphi(Y) + [\varphi(X), \varphi(Y)]/2 + \cdots) \\ &= \exp(\varphi(X))\exp(\varphi(Y)) = \Phi(\exp(X))\Phi(\exp(Y)),\end{aligned}$$

   by Campbell-Baker-Hausdorff and the fact that the omitted terms are all Lie brackets, which are preserved by φ. Hence Φ is a Lie group homomorphism in U.
2. Let $A \in G$. Pick a path from A to $\mathbf{1}$ in G. The open covering of this path by δ-balls has a finite subcover; hence there exists a sequence $\mathbf{1} = A_1, A_2, \ldots, A_m = A$ with

$$A_1, A_1^{-1}A_2, \ldots, A_{m-1}^{-1}A_m \in U.$$

   We now define $\Phi(A) = \Phi(A_1)\Phi(A_1^{-1}A_2)\cdots\Phi(A_{m-1}^{-1}A_m)$.
3. Check that refining the sequence $A_1, \ldots, A_m$ by inserting extra points does not affect the value of $\Phi(A)$. Hence given a path from A to $\mathbf{1}$, $\Phi(A)$ is independent of the choice of sequence of points along the path.

4. For any two paths p, q from A to $\mathbf{1}$, there is a sequence of "elementary deformations" $p = p_1, p_2, \ldots, p_n = q$ such that for each deformation from p_i to p_{i+1}, no point on the path moves by more than δ. (This follows from compactness of $[0,1] \times [0,1]$.)
5. Check that $\Phi(A)$ is constant under elementary deformations, so $\Phi(A)$ is independent of the choice of path from A to $\mathbf{1}$. Hence $\Phi : G \to H$ is well-defined.
6. Check that Φ is a Lie algebra homomorphism that induces φ. □

Corollary. *Let* G, H *are simply connected matrix Lie groups. If* $\mathfrak{g}, \mathfrak{h}$ *are isomorphic, then* G, H *are isomorphic.*

Proof. Let $\varphi : \mathfrak{g} \to \mathfrak{h}$ be an isomorphism, so there exists homomorphisms $\Phi : G \to H$ inducing φ and $\Psi : H \to G$ inducing φ^{-1}. Now $\Psi \circ \Phi : G \to G$ is the unique homomorphism that induces $\varphi^{-1} \circ \varphi = \mathrm{id}_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}$, so $\Psi = \Phi^{-1}$. Hence Φ is a Lie isomorphism between G and H. □

Introduction to general Lie groups

We have seen how the Lie bracket and the exponential map shed light on the structure of matrix Lie groups. However, their definitions rely on the multiplication in the tangent space, which is not usually defined in the general case. In this section, we briefly sketch (without proofs) how to extend these constructions to general Lie groups.

As usual, for a Lie group G, it is customary to write $\mathfrak{g} = T_{1_G} G$.

The exponential map For a matrix Lie group G, exp satisfies the differential equation $\frac{d}{dt} \exp(tX) = X \exp(tX)$ ($X \in \mathfrak{g}$), which can be seen as an integral curve of the vector field $V(g) = gX$ in $GL(n, \mathbb{C})$. In fact, V is a vector field on G due to the following:

Proposition. *For any* $g \in G$, *we have* $T_g G = gT_{\mathbf{1}} G$.

Proof. It suffices to consider the bijection between smooth paths A through $\mathbf{1}$ and smooth paths B through g given by $B(t) = gA(t)$. □

Hence $V(g) = gX \in gT_{\mathbf{1}} G = T_g G$, so V is a vector field on G.

Note that V satisfies the special property that $g_1 V(g_2) = g_1 g_2 X = V(g_1 g_2)$ for all $g_1, g_2 \in G$. Such a vector field is called *left-invariant*.

Proposition. *Let* G *be a general Lie group. Then for all* $X \in \mathfrak{g}$, *there exists a unique smooth left-invariant vector field* V_X *on* G *with* $V_X(1_G) = X$.

Hence there is a unique smooth path $\gamma_X : (-\varepsilon, \varepsilon) \to G$ which is the integral curve of V_X, satisfying $\gamma_X(0) = 1_G$. This integral curve can be extended by the identity $\gamma_X(t + t') = \gamma_X(t)\gamma_X(t')$, so we may now define the exponential map:

$$\exp : \mathfrak{g} \to G, \qquad \exp(X) \coloneqq \gamma_X(1).$$

This is not a perfect analog of the matrix exponential; in particular, there is no identity of the form $\exp(X)\exp(Y) = \exp(X+Y)$, because it doesn't make sense to say that $X, Y \in \mathfrak{g}$ commute (since $\mathfrak{g}$ does not have multiplication)! However, the following properties of exp still hold.

Proposition. *Let G be a Lie group, and let $X \in \mathfrak{g}$. Then:*

(a) $\exp(X) = 1 + X + \cdots$*, ie.* $\exp(0) = 1_G$ *and* $(T\exp)_0 = \mathrm{id}_{\mathfrak{g}}$.

(b) exp *is a diffeomorphism between a neighbourhood of 0 in* $\mathfrak{g}$ *and a neighbourhood of* 1_G *in* G.

The Lie bracket By the above, exp has a local inverse near 0, which we denote by log. Hence for sufficiently small $X, Y \in \mathfrak{g}$, $\exp(X)\exp(Y)$ will be close to 1_G, so that

$$\exp(X)\exp(Y) = \exp(\mu(X, Y))$$

for some unique μ defined near $(0, 0)$ in $\mathfrak{g} \times \mathfrak{g}$.

Proposition. *The Taylor series for* μ *is given by* $\mu(X, Y) = X + Y + \frac{[X,Y]}{2} + \cdots$*, where the dots represent terms of order at least 3, and* $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *is a bilinear map satisfying*

$$[X, Y] = -[Y, X], \qquad [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

Hence $\mathfrak{g}$, equipped with the Lie bracket, is a Lie algebra, denoted by $\mathfrak{g} = \mathrm{Lie}(G)$.

Fundamental theorems of Lie theory A central idea in the study of Lie groups is that the tangent space at the identity "almost" determines the entire structure of the Lie group, which we have seen in action in the case of matrix Lie groups. We leave the reader with more examples of the interplay between algebra and topology, the essence of Lie theory, with a sampling of key theorems.

Theorem. *For any Lie group* G*, there is a bijection between connected Lie subgroups* $H \subseteq G$ *and Lie subalgebras* $\mathfrak{h} \subseteq \mathfrak{g}$*, given by* $H \mapsto \mathfrak{h} = \mathrm{Lie}(H)$.

Theorem. *If* G_1, G_2 *are Lie groups with* G_1 *connected and simply connected, then every Lie algebra homomorphism* $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ *determines a Lie group homomorphism* $\overline{\varphi} : G_1 \to G_2$ *which induces* φ.

Theorem. *For any finite-dimensional Lie algebra* $\mathfrak{g}$*, there is a simply connected Lie group* G*, unique up to isomorphism, such that* $\mathrm{Lie}(G) = \mathfrak{g}$.

Moreover, any other connected Lie group G' *with* $\mathrm{Lie}(G') = \mathfrak{g}$ *is of the form* $G' \cong G/Z$ *for some discrete central subgroup* $Z \subseteq G$.

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