# Approximate Gaussian Elimination for Laplacian Systems

MA4291 Presentation

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# Laplacian matrices

**Definition**

A *Laplacian matrix* is a symmetric matrix $\mathbf{L}$ such that:

- $L_{ii} \geq 0$, and $L_{ij} \leq 0$ for $i \neq j$; and
- the entries in each row sum to 0.

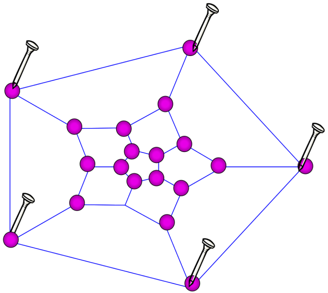
| Laplacians | | Weighted graphs |
| --- | --- | --- |
| $\mathbf{L} \in \mathbb{R}^{n \times n}$ | $\longleftrightarrow$ | $G = ([n], E)$, $w : E \to \mathbb{R}_{\geq 0}$ |

$$\mathbf{L} = \sum_{i,j} w_{ij} \times \begin{array}{c c} & \begin{array}{cc} i & j \end{array} \\ \begin{array}{c} i \\ j \end{array} & \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix} \end{array}$$

$$= \sum_{i,j} w_{ij} (\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^T$$

$$\mathbf{x}^T \mathbf{L} \mathbf{x} = \sum_{i,j} w_{ij} (x_i - x_j)^2$$

# Example: springy graphs



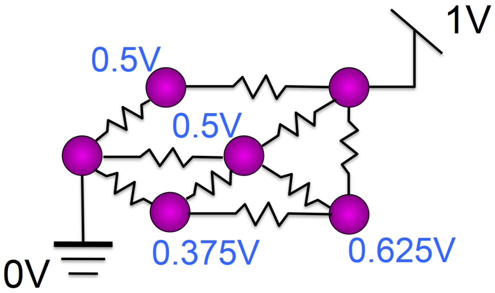
Vertex positions $(x_i, y_i)$, spring constants $w_{ij}$

Elastic potential energy: $\frac{1}{2}\sum_{i,j} w_{ij}(x_i - x_j)^2$

Energy minimised when

$$x_i = \frac{\sum_{j\neq i} w_{ij}x_j}{\sum_j w_{ij}} \implies \mathbf{Lx} = \mathbf{b}$$

# Example: resistor networks



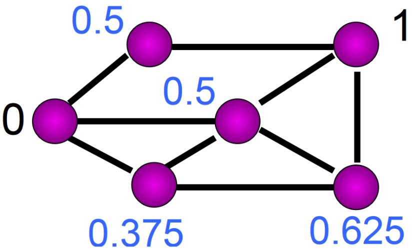

Voltages $x_i$, resistances $w_{ij}^{-1}$

Voltages minimise power dissipated $\frac{1}{2}\sum_{i,j} w_{ij}(x_i - x_j)^2$

(Dirichlet Principle)

$$x_i = \frac{\sum_{j \neq i} w_{ij} x_j}{\sum_j w_{ij}} \implies \mathbf{Lx} = \mathbf{b}$$

# Example: learning on graphs




Function values $x_i$, edge weights $w_{ij}$

Minimise $\frac{1}{2}\sum_{i,j} w_{ij}(x_i - x_j)^2$

$$x_i = \frac{\sum_{j \neq i} w_{ij} x_j}{\sum_j w_{ij}} \implies \mathbf{L}\mathbf{x} = \mathbf{b}$$

# The main problem

**Main problem**

Given Laplacian $\mathbf{L} \in \mathbb{R}^{n \times n}$ with $m$ nonzero entries, and $\mathbf{b} \in \mathbb{R}^n$, efficiently solve $\mathbf{Lx} = \mathbf{b}$ to accuracy $\varepsilon$.

LU/Cholesky factorisation: $O(n^3)$, even for sparse $\mathbf{L}$

Spielman-Teng 2004: $O(m \ln^{50} n \ln(1/\varepsilon))$

Cohen et al. 2014: $O(m \ln^{1/2} n \ln(1/\varepsilon))$

Kyng 2017: $O(m \ln^2 n \ln(1/\varepsilon))$, simple

# Linear iteration

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$
$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Iterative method:
$$\mathbf{x}_{k+1} = \mathbf{M}\mathbf{x}_k + \mathbf{N}\mathbf{b}$$
$$\mathbf{x}_* = \mathbf{M}\mathbf{x}_* + \mathbf{N}\mathbf{b}$$

Convergence:
$$\mathbf{e}_{k+1} = \mathbf{M}\mathbf{e}_k$$
$$\implies \|\mathbf{M}\| < 1$$

Consistency:
$$\mathbf{x} = \mathbf{x}_* = (\mathbf{I} - \mathbf{M})^{-1}\mathbf{N}\mathbf{b}$$
$$\implies \mathbf{M} = \mathbf{I} - \mathbf{N}\mathbf{A}$$

**Definition**

$$\mathbf{B} \approx_c \mathbf{A} \iff \|\mathbf{B}^{-1}\mathbf{A} - \mathbf{I}\| \leq c$$

Want: approximate Cholesky factorisation

$\mathbf{U}$ sparse upper triangular, $\mathbf{U}^T\mathbf{U} \approx_{1/2} \mathbf{L}$

# Cholesky factorisation

$$\begin{pmatrix} 16 & -4 & -8 & -4 \\ -4 & 5 & 0 & -1 \\ -8 & 0 & 9 & -1 \\ -4 & -1 & -1 & 6 \end{pmatrix} = \mathbf{c}_1\mathbf{c}_1^T + \begin{pmatrix} \phantom{0} & & & \\ & 4 & -2 & -2 \\ & -2 & 5 & -3 \\ & -2 & -3 & 5 \end{pmatrix} \quad \mathbf{c}_1 = \begin{pmatrix} 4 \\ -1 \\ -2 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -2 & -2 \\ -2 & 5 & -3 \\ -2 & -3 & 5 \end{pmatrix} = \mathbf{c}_2\mathbf{c}_2^T + \begin{pmatrix} \phantom{0} & & \\ & 4 & -4 \\ & -4 & 4 \end{pmatrix} \qquad \mathbf{c}_2 = \begin{pmatrix} 0 \\ 2 \\ -1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix} = \mathbf{c}_3\mathbf{c}_3^T \qquad \mathbf{c}_3 = \begin{pmatrix} 0 \\ 0 \\ 2 \\ -2 \end{pmatrix}$$

# Cholesky factorisation

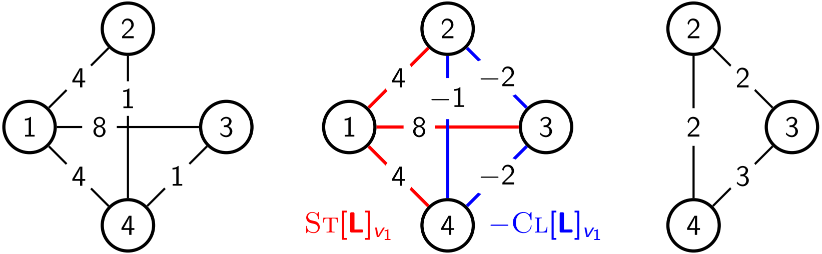
$$\begin{pmatrix} 16 & -4 & -8 & -4 \\ -4 & 5 & 0 & -1 \\ -8 & 0 & 9 & -1 \\ -4 & -1 & -1 & 6 \end{pmatrix} = \begin{pmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{0} \end{pmatrix}^T$$

$$= \mathbf{U}^T \mathbf{U}$$

$$\mathbf{U} = \begin{pmatrix} 4 & -1 & -2 & -1 \\ & 2 & -1 & -1 \\ & & 2 & -1 \\ & & & 0 \end{pmatrix}$$

# Cholesky factorisation

$$\begin{pmatrix} 16 & -4 & -8 & -4 \\ -4 & 5 & 0 & -1 \\ -8 & 0 & 9 & -1 \\ -4 & -1 & -1 & 6 \end{pmatrix} = \begin{pmatrix} 16 & -4 & -8 & -4 \\ -4 & 1 & 2 & 1 \\ -8 & 2 & 4 & 2 \\ -4 & 1 & 2 & 1 \end{pmatrix} + \begin{pmatrix} & & & \\ & 4 & -2 & -2 \\ & -2 & 5 & -3 \\ & -2 & -3 & 5 \end{pmatrix}$$




Remove one vertex, add a clique to its neighbours

# The algorithm

$\textsc{CholFac}(\mathbf{L})$

Split each edge into $\rho = 12 \ln^2(1/\delta)$ copies
Pick random permutation of vertices $v_1, \ldots, v_n$
$\mathbf{S}_0 \leftarrow \mathbf{L}$
**for** $i = 1, \ldots, n$ **do**

$$\mathbf{c}_i \leftarrow \begin{cases} \frac{1}{\sqrt{\mathbf{S}_{i-1}(v_i, v_i)}} \mathbf{S}_{i-1}(v_i, :) & \text{if } \mathbf{S}_{i-1}(v_i, v_i) \neq 0 \\ \mathbf{0} & \text{else} \end{cases}$$

$\quad \mathbf{C}_i \leftarrow \mathrm{CL}[\mathbf{S}_{i-1}]_{v_i}$
$\quad \mathbf{S}_i \leftarrow \mathbf{S}_{i-1} - \mathrm{ST}[\mathbf{S}_{i-1}]_{v_i} + \mathbf{C}_i$
**end for**
$\mathbf{U} \leftarrow \begin{pmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_n \end{pmatrix}^T$
**return U**

$\text{CholApx}'(\mathbf{L}, \delta)$

Split each edge into $\rho = 12 \ln^2(1/\delta)$ copies
Pick random permutation of vertices $v_1, \ldots, v_n$
$\mathbf{S}_0 \leftarrow \mathbf{L}$
**for** $i = 1, \ldots, n$ **do**

$$\mathbf{c}_i \leftarrow \begin{cases} \frac{1}{\sqrt{\mathbf{S}_{i-1}(v_i, v_i)}} \mathbf{S}_{i-1}(v_i, :) & \text{if } \mathbf{S}_{i-1}(v_i, v_i) \neq 0 \\ \mathbf{0} & \text{else} \end{cases}$$

$\mathbf{C}_i \leftarrow \text{CliqueSample}(\mathbf{S}_{i-1}, v_i)$
$\mathbf{S}_i \leftarrow \mathbf{S}_{i-1} - \text{St}[\mathbf{S}_{i-1}]_{v_i} + \mathbf{C}_i$
**end for**
$\mathbf{U} \leftarrow \begin{pmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_n \end{pmatrix}^T$
**return** $\mathbf{U}$

Idea: add $\deg(v)$ random edges instead of $\binom{\deg(v)}{2}$

# The algorithm

$\textsc{CliqueSample}(\mathbf{S}, v)$

**for** $e = 1, \ldots, \deg_{\mathbf{S}}(v)$ **do**
  Sample $(v, u_1)$ with probability $w(v, u_1)/w_{\mathbf{S}}(v)$
  Sample $(v, u_2)$ uniformly
  $\mathbf{Y}_e \leftarrow \frac{w(v,u_1)w(v,u_2)}{w(v,u_1)+w(v,u_2)}(\mathbf{e}_{u_1} - \mathbf{e}_{u_2})(\mathbf{e}_{u_1} - \mathbf{e}_{u_2})^T$
**end for**
**return** $\sum_e \mathbf{Y}_e$ ▷ expected value $= \textsc{Cl}[\mathbf{S}]_v$

**Theorem (Kyng 2017)**

$\textsc{CholApx}'(\mathbf{L}, \delta)$ $(\delta < n^{-100})$ *returns* $\mathbf{U}$ *with* $O(m \ln^2(1/\delta) \ln n)$ *nonzero entries such that* $\mathbf{U}^T\mathbf{U} \approx_{1/2} \mathbf{L}$ *with probability* $1 - O(\delta)$. *Moreover,* $\textsc{CholApx}'(\mathbf{L}, \delta)$ *runs in* $O(tm \ln^2(1/\delta) \ln n)$ *time with probability* $1 - n^{-t}$ *for* $t > 1$.

# The matrix martingale

$$\begin{aligned}
&\mathbf{c}_i \leftarrow \frac{1}{\sqrt{\mathbf{S}_{i-1}(v_i,v_i)}}\mathbf{S}_{i-1}(v_i,:) \quad \text{if } \mathbf{S}_{i-1}(v_i,v_i) \neq 0 \\
&\mathbf{C}_i \leftarrow \textsc{CliqueSample}(\mathbf{S}_{i-1}, v_i) \\
&\mathbf{S}_i \leftarrow \mathbf{S}_{i-1} - \textsc{St}[\mathbf{S}_{i-1}]_{v_i} + \mathbf{C}_i
\end{aligned}$$

Let $\mathbf{L}_i = \mathbf{S}_i + \sum_{j=1}^{i} \mathbf{c}_j\mathbf{c}_j^T$. Note that $\mathbf{L}_n = \mathbf{U}^T\mathbf{U}$.

$$\begin{aligned}
\mathbf{L}_i - \mathbf{L}_{i-1} &= \mathbf{S}_i - \mathbf{S}_{i-1} + \mathbf{c}_i\mathbf{c}_i^T \\
&= \mathbf{C}_i - \textsc{St}[\mathbf{S}_{i-1}]_{v_i} + \mathbf{c}_i\mathbf{c}_i^T \\
&= \mathbf{C}_i - \textsc{Cl}[\mathbf{S}_{i-1}]_{v_i}
\end{aligned}$$

$(\mathbf{L}_i - \mathbf{L})_{i=0}^{n}$ is a zero-mean martingale!

We want to bound

$$\mathbb{P}\left(\left\|\mathbf{L}^{-1}\mathbf{L}_n - \mathbf{I}\right\| \geq \frac{1}{2}\right) = \mathbb{P}\left(\left\|\overline{\mathbf{L}_n} - \mathbf{I}\right\| \geq \frac{1}{2}\right),$$

where $\overline{\mathbf{A}} := \mathbf{L}^{-1/2}\mathbf{A}\mathbf{L}^{-1/2}$ if $\mathbf{A}$ symmetric with $\ker(\mathbf{L}) \subseteq \ker(\mathbf{A})$.

# First try

$$\mathbf{Y}_{i,e} \leftarrow \frac{w(v,u_1)w(v,u_2)}{w(v,u_1)+w(v,u_2)}(\mathbf{e}_{u_1} - \mathbf{e}_{u_2})(\mathbf{e}_{u_1} - \mathbf{e}_{u_2})^T$$

Let $\mathbf{X}_{i,e} = \mathbf{Y}_{i,e} - \mathbb{E}_{<(i,e)}\mathbf{Y}_{i,e}$:

$$\mathbf{L}_i - \mathbf{L}_{i-1} = \mathbf{C}_i - \mathrm{CL}[\mathbf{S}_{i-1}]_{v_i} = \textstyle\sum_e \mathbf{X}_{i,e}.$$

Can show $\|\overline{\mathbf{Y}_{i,e}}\| \leq 1/\rho$, so

$$\|\overline{\mathbf{X}_{i,e}}\| \leq \max(\|\overline{\mathbf{Y}_{i,e}}\|, \|\mathbb{E}_{<(i,e)}\overline{\mathbf{Y}_{i,e}}\|) \leq 1/\rho.$$

Matrix Azuma: martingale length $\sim m\rho[\frac{1}{n} + \cdots + \frac{1}{2} + 1] \sim m\rho \ln n$,

$$\begin{aligned} \mathbb{P}\left(\left\|\overline{\mathbf{L}_n} - \mathbf{I}\right\| \geq t\right) &\leq \exp\left(\frac{-t^2}{8\sum 1/\rho^2}\right) \\ &\leq \exp\left(\frac{-Kt^2}{m\rho \ln n/\rho^2}\right) \\ &= \exp\left(\frac{-K' t^2 \ln n}{m}\right) \end{aligned}$$

# Second try

$$\mathbf{Y}_{i,e} \leftarrow \frac{w(v,u_1)w(v,u_2)}{w(v,u_1)+w(v,u_2)}(\mathbf{e}_{u_1} - \mathbf{e}_{u_2})(\mathbf{e}_{u_1} - \mathbf{e}_{u_2})^T$$

$$\mathbf{X}_{i,e} = \mathbf{Y}_{i,e} - \mathbb{E}_{<(i,e)}\mathbf{Y}_{i,e}$$

$$\mathbf{L}_i - \mathbf{L}_{i-1} = \sum_e \mathbf{X}_{i,e}$$

$$\|\overline{\mathbf{X}_{i,e}}\| \leq 1/\rho$$

Define a new martingale with stopping condition:

$$\mathbf{Z}_{i,e} = \begin{cases} \overline{\mathbf{X}_{i,e}} & \text{if } \|\sum_{j<i}\sum_f \mathbf{Z}_{j,f}\| \leq \frac{1}{2} \\ \mathbf{0} & \text{else} \end{cases}$$

$$\mathbf{T}_{i,e} = \sum_{(j,f)\leq(i,e)} \mathbf{Z}_{j,f}$$

Then

$$\mathbb{P}\left(\left\|\overline{\mathbf{L}_n} - \mathbf{I}\right\| \geq \frac{1}{2}\right) \leq \mathbb{P}\left(\exists(i,e) \,:\, \|\mathbf{T}_{i,e}\| \geq \frac{1}{2}\right).$$

# Matrix Freedman inequality

**Theorem (Freedman 1975)**

*Let $(A_k)_{k\geq 0}$ be a real-valued martingale with $B_k \leq R$, where $B_k = A_k - A_{k-1}$. Let $W_k = \sum_{j=1}^{k} \mathbb{E}_{<j}(B_j^2)$. Then for all $t \geq 0$ and $\sigma^2 > 0$,*

$$\mathbb{P}\left(\exists k \,:\, A_k \geq t \text{ and } W_k \leq \sigma^2\right) \leq \exp\left(\frac{-t^2/2}{\sigma^2 + Rt/3}\right).$$

**Theorem (Tropp 2011)**

*Let $(\mathbf{A}_k)_{k\geq 0}$ be a symmetric $d \times d$-matrix martingale with $\lambda_{max}(\mathbf{B}_k) \leq R$, where $\mathbf{B}_k = \mathbf{A}_k - \mathbf{A}_{k-1}$. Let $\mathbf{W}_k = \sum_{j=1}^{k} \mathbb{E}_{<j}(\mathbf{B}_j^2)$. Then for all $t \geq 0$ and $\sigma^2 > 0$,*

$$\mathbb{P}\left(\exists k \,:\, \lambda_{max}(\mathbf{A}_k) \geq t \text{ and } \|\mathbf{W}_k\| \leq \sigma^2\right) \leq 2d \exp\left(\frac{-t^2/2}{\sigma^2 + Rt/3}\right).$$

# Freedman inequality: proof

The function $g(x) = \dfrac{e^x - 1 - x}{x^2}$ is increasing.

$e^{\lambda x} \leq 1 + \lambda x + (e^\lambda - 1 - \lambda)x^2$ for all $\lambda \geq 0$, $x \leq 1$.

Let $B$ be a r.v. with $B \leq 1$, $\mathbb{E}B = 0$. Then

$$\begin{aligned}\mathbb{E}\exp(\lambda B) &\leq 1 + (e^\lambda - 1 - \lambda)\mathbb{E}(B^2) \\ &\leq \exp((e^\lambda - 1 - \lambda)\mathbb{E}(B^2)).\end{aligned}$$

Take $B = B_n$:

$$\begin{aligned}&\exp(\lambda A_{k-1} - (e^\lambda - 1 - \lambda)W_{k-1}) \\ &\geq \mathbb{E}_{<k}\exp(\lambda(A_{k-1} + B_k) - (e^\lambda - 1 - \lambda)(W_{k-1} + \mathbb{E}_{<k}(B_k^2))) \\ &= \mathbb{E}_{<k}\exp(\lambda A_k - (e^\lambda - 1 - \lambda)W_k)\end{aligned}$$

Hence $\big(\exp(\lambda A_k - (e^\lambda - 1 - \lambda)W_k)\big)_k$ is a supermartingale.

# Freedman inequality: proof

Let $s = \min\{k : A_k \geq t\}$ if defined, $\infty$ otherwise. Then

$$\mathbb{E}\left(\mathbb{1}_{s<\infty} \exp(\lambda A_s - (e^\lambda - 1 - \lambda)W_s)\right) \leq 1.$$

Let $E$ be the event that $A_k \geq t$ and $W_k \leq \sigma^2$ for some $k$.
Now $s < \infty$ on $E$, so

$$\begin{aligned} 1 &\geq \mathbb{E}\left(\mathbb{1}_E \exp(\lambda A_s - (e^\lambda - 1 - \lambda)W_s)\right) \\ &\geq \mathbb{P}(E) \exp(\lambda t - (e^\lambda - 1 - \lambda)\sigma^2) \\ \mathbb{P}(E) &\leq \exp(-\lambda t + (e^\lambda - 1 - \lambda)\sigma^2). \end{aligned}$$

Use $e^\lambda - 1 - \lambda \leq \frac{\lambda^2}{2!}\left(1 + \frac{t}{3} + (\frac{t}{3})^2 + \cdots\right) = \frac{\lambda^2/2}{1-\lambda/3}$, $\lambda = \frac{t}{\sigma^2 + t/3}$:

$$\mathbb{P}(E) \leq \exp\left(\frac{-t^2/2}{\sigma^2 + t/3}\right).$$

If $\mathbf{B}$ is a symmetric r.v. with $\mathbb{E}\mathbf{B} = \mathbf{0}$ and $\lambda_{\max}(\mathbf{B}) \leq 1$, then

$$\mathbb{E}\exp(\theta\mathbf{B}) \preccurlyeq \exp\left((e^\theta - \theta - 1)\mathbb{E}(\mathbf{B}^2)\right).$$

Let $S_k(\theta) := \operatorname{Tr}\exp(\theta\mathbf{A}_k - (e^\theta - \theta - 1)\mathbf{W}_k)$. Then $S_0 = d$, and

$$\begin{aligned}
S_{k-1} &= \operatorname{Tr}\exp(\theta\mathbf{A}_{k-1} - (e^\theta - \theta - 1)\mathbf{W}_{k-1}) \\
&= \operatorname{Tr}\exp(\theta\mathbf{A}_{k-1} - (e^\theta - \theta - 1)(\mathbf{W}_k - \mathbb{E}_{<k}(\mathbf{B}_j^2))) \\
&\geq \operatorname{Tr}\exp(\theta\mathbf{A}_{k-1} - (e^\theta - \theta - 1)\mathbf{W}_k + \ln\mathbb{E}_{<k}\exp(\theta\mathbf{B}_j)) \\
&\overset{*}{\geq} \mathbb{E}_{<k}\operatorname{Tr}\exp(\theta\mathbf{A}_{k-1} - (e^\theta - \theta - 1)\mathbf{W}_k + \theta\mathbf{B}_k) \\
&= \mathbb{E}_{<k}S_k.
\end{aligned}$$

**Theorem (Lieb 1973)**

*If $\mathbf{H}$ is symmetric, then $\mathbf{A} \mapsto \operatorname{Tr}\exp(\mathbf{H} + \ln\mathbf{A})$ is concave on the set of positive definite matrices.*

Hence $(S_k)_k$ is a supermartingale, finish as before.

# Main theorem: proof sketch

$$\|\mathbf{Z}_{i,e}\| \leq 1/\rho$$
$$\mathbf{T}_{i,e} = \sum_{(j,f)\leq(i,e)} \mathbf{Z}_{j,f}$$

Let $\mathbf{W}_{i,e} = \sum_{(j,f)\leq(i,e)} \mathbb{E}_{<(j,f)}(\overline{\mathbf{Z}_{j,f}}^2)$. Then

$$\mathbb{P}\left(\exists(i,e) : \|\mathbf{T}_{i,e}\| \geq \frac{1}{2}\right) \leq \mathbb{P}\left(\exists(i,e) : \|\mathbf{T}_{i,e}\| \geq \frac{1}{2} \text{ and } \|\mathbf{W}_{i,e}\| \leq \sigma^2\right) + \mathbb{P}\left(\exists(i,e) : \|\mathbf{W}_{i,e}\| \geq \sigma^2\right)$$

First term: Matrix Freedman ($\sigma^2 = \ln(1/\delta)/\rho$, $R = 1/\rho$, $t = 1/2$)

$$\mathbb{P}_1 \leq n \exp\left(\frac{-t^2/2}{\sigma^2 + Rt/3}\right) \leq 2\delta.$$

Second term: Matrix Freedman

$$\mathbf{W}_i = \mathbf{W}_{i,e_{\text{last}}} \qquad \mathbf{V}_i = \mathbf{W}_i - \mathbb{E}_{<i}\mathbf{W}_i$$
$$\mathbf{R}_i = \sum_{j=1}^{i} \mathbf{V}_j \qquad \mathbf{M}_i = \sum_{j=1}^{i} \mathbb{E}_{<j}\mathbf{V}_j^2$$

# Extensions

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$\textsc{CholApx}(\mathbf{L}, \delta)$

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. . .

Pick $v_i$ uniformly from vertices with at most twice average degree

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**Theorem (Kyng 2017)**

$\textsc{CholApx}(\mathbf{L}, \delta)$ $(\delta < n^{-100})$ *returns* $\mathbf{U}$ *with* $O(m \ln^2(1/\delta) \ln n)$ *nonzero entries such that* $\mathbf{U}^T\mathbf{U} \approx_{1/2} \mathbf{L}$ *with probability* $1 - O(\delta)$. *Moreover,* $\textsc{CholApx}(\mathbf{L}, \delta)$ *runs in* $O(m \ln^2(1/\delta) \ln n)$ *time.*

Random Gaussian row elimination

- Consistent runtimes for different graph families
- Open problem: why does this work?

# References


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