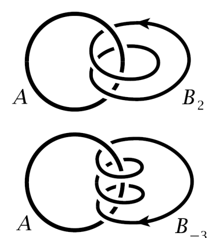
MA4266 Review Report

Ang Yan Sheng
A0144836Y

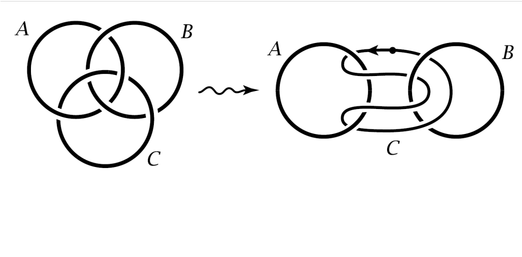
Allen Hatcher's *Algebraic Topology*, published in 2002, is an introductory graduate-level text on algebraic topology from a classical viewpoint, with "a general slant towards geometric, rather than algebraic, aspects of the subject" (ix). The text covers the general theory of fundamental groups, covering spaces, homology, cohomology, and homotopy theory. Due to the lack of time to pursue all of these topics in depth, I will limit the scope of this review to the first chapter on Fundamental Groups and Covering Spaces (21–82), focusing on the first two sections up to van Kampen's theorem.

To me, a mathematics text should not only present the important results in a field, but also explain why the result should be true, or why it is important. In this regard, Hatcher manages to clearly provide both motivation and intuition behind each new concept. From the start of the chapter, Hatcher describes the study of algebraic topology as "techniques for forming algebraic images of topological spaces," so as to "reconstruct accurately the shapes of... large and interesting classes of spaces" (21). This motivates the study of the fundamental group, the simplest such algebraic image.

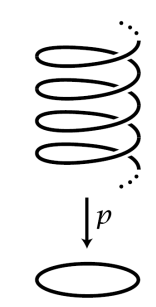


Before delving into formal definitions, Hatcher first explores the idea of the fundamental group (21–24). Given two loops in $\mathbb{R}^3$, we can 'measure' how they are linked together by the 'winding number,' which is an integer (see right figure). Also, the group structure on $\mathbb{Z}$ suggests a natural addition operation on loops.

However, using an abelian group like $\mathbb{Z}$ is insufficient, as the example of three circles in the Borromean rings illustrates (see left figure). Through a series of further motivating examples, Hatcher arrives at a tentative definition of this group, and explains its significance: the fundamental group can be used "to tell when a pair of circles A and B is linked," but much more generally, to "show that two spaces are not homeomorphic by showing their fundamental groups are not isomorphic" (24).



To a reader like myself who might be meeting these concepts for the first time, this approach is a lot more valuable than the usual definition-theorem presentation of many math texts, as it tries to convey the mental picture that one should have when thinking about such objects. Hatcher also includes numerous pictures to aid in his explanations, which helps me build up geometric intuition. The text also tells me about *why* I should care about constructions such as the fundamental group, putting it in context of the bigger problem of classifying topological spaces.



Section 1.1, Basic Constructions, goes through the usual definitions of paths, homotopies, and the fundamental group. To prove that the fundamental group of the circle is $\mathbb{Z}$, Hatcher introduces the notion of covering spaces (29–31; see figure), which is covered in greater depth in Section 1.3. Though this might seem redundant, I appreciate that this specific example is covered in detail here, so that I have a concrete example to anchor myself to when approaching the general theory later.

After the technical proof of $\pi_1(S^1) \cong \mathbb{Z}$, Hatcher immediately presents several interesting applications, proving the Fundamental Theorem of Algebra (31), the Brouwer fixed point theorem in $\mathbb{R}^2$ (31–32), and the Borsuk-Ulam theorem on S^2 (32–33), illustrating the power of the fundamental group as a new topological tool. The section concludes with some results about how the fundamental group behaves under homeomorphisms: if X retracts to a subspace A, then the homomorphism $\pi_1(A) \to \pi_1(X)$ induced by the inclusion $A \hookrightarrow X$ is injective, and if X deformation retracts to A then this is an isomorphism.

Section 1.2, Van Kampen's Theorem (40–55), deals with computing the fundamental group of a space given a decomposition into 'simpler' pieces with known fundamental groups. As I understand from Hatcher's overview, if A is the union of subspaces A_α, then $\pi_1(A)$ is isomorphic to the free product of the $\pi_1(A_\alpha)$, after taking the quotient of the subgroup generated by all relations induced by the fact that

$$\begin{array}{ccccc} & & \pi_1(A_\alpha) & & \\ & \nearrow & & \searrow & \\ \pi_1(A_\alpha \cap A_\beta) & & & & \pi_1(A) \\ & \searrow & & \nearrow & \\ & & \pi_1(A_\beta) & & \end{array}$$

commutes. To me, this intuitively means that we get $\pi_1(A)$ simply by 'gluing' the groups $\pi_1(A_\alpha)$ together 'along' the $\pi_1(A_\alpha \cap A_\beta)$, which agrees well with the first example he gives in this section (two circles joined together at a point).

I have seen texts where van Kampen's theorem is only proved after developing the theory of triangulations; or the proof is relegated to an appendix. Hence I think it is commendable that Hatcher tackles the proof at such an early stage in the book. The method of proof, by splitting $[0,1] \times [0,1]$ into small rectangles, reminds me of a proof in complex analysis of the fact that a region $R \subseteq \mathbb{C}$ is simply connected if and only if $\oint_\gamma \frac{dz}{z-a} = 0$ for all closed loops γ in R and $a \notin R$; seeing this link allowed me to understand this proof at a deeper level.

All in all, Hatcher's text features clear exposition, provides strong motivation for each new definition, and includes pictures which enhance the geometric intuition beneath the algebraic manipulations. Thus, I believe that this text is uniquely suited to self-study for motivated students looking for an introduction to algebraic topology, or as a supplementary text for students in a first course to algebraic topology.

References

A. Hatcher, *Algebraic Topology*, Cambridge Univ. Press, 2002. Accessed 2 April 2017 from `http://www.math.cornell.edu/~hatcher/AT/ATch1.pdf`.