MA4266 Reading Report

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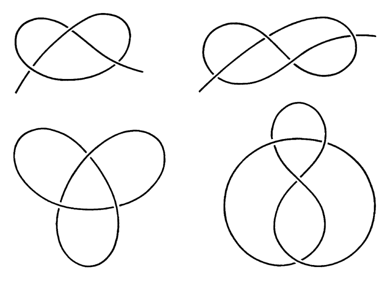
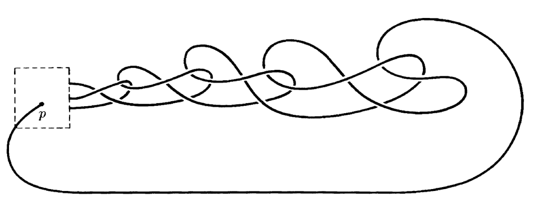
The central driving force behind topology are questions of classification. What are the different possible topological spaces (satisfying certain conditions)? How do we distinguish them? The study of even the easiest cases of such questions has given us powerful algebraic tools to tell different objects apart; the classification of closed surfaces by Euler characteristic is a familiar example.

In the same spirit, we may ask: What are the different ways of imbedding X into Y? One of the simplest interesting cases, where $X = S^1$ and $Y = \mathbb{R}^3$, gives rise to the study of *knot theory*, the subject of the 1963 text *Introduction to Knot Theory* by Richard Crowell and Ralph Fox. In this report, I will examine the ideas in topology that have contributed to progress towards this classification question.

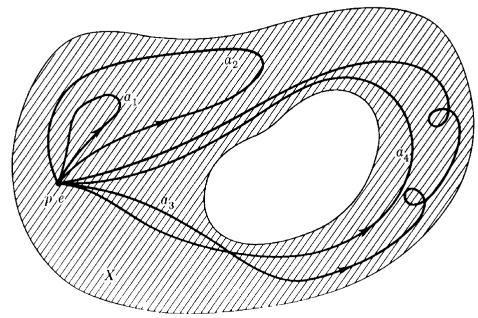


Chapter 1 is an introduction to knots and knot types. Firstly, in order to turn the idea of a knot into a topological object, we must define what it means for two knots to be equivalent to each other. This is hard if the ends of the knot are free (top), so we close up the knot into a loop (bottom).

Now intuitively we know that the knot on the left (the *trefoil knot*) cannot be physically deformed into the knot on the right (the *figure-eight knot*), and neither can be deformed into a flat circle (the *unknot*). Since all knots are homeomorphic to S^1, and thus to each other, we need a stronger concept of knot equivalence. This is achieved by stipulating that two knots K_1, K_2 are equivalent if one can be *isotopically deformed* into the other, ie. there exists a family of homeomorphisms $h_t : \mathbb{R}^3 \to \mathbb{R}^3$, $t \in [0,1]$, such that $h_0 = \mathrm{id}_{\mathbb{R}^3}$, $H(t,p) = h_t(p)$ is continuous in both t and p, and $h_1(K_1) = K_2$. Crowell and Fox state that this is equivalent to the existence of an orientation-preserving homeomorphism mapping K_1 to K_2, though rigorous treatment of this idea requires homology theory and is omitted.



To rule out pathological possibilities, we usually restrict our attention to tame knots, ie. those that are equivalent to a polygonal knot. For example, the knot pictured on the left is smooth everywhere except at p, and can "clearly" be untied by pulling on loops from the right. However, it can be shown to be not tame in the above sense, and will be excluded from our discussion. In fact, all curves which are continuously differentiable in the arc length parametrisation are tame, so we don't usually run into "wild" knots like the one pictured.



In Chapter 2, we are introduced to the concept of the fundamental group of a space as an important topological invariant. For a fixed point $p \in X$, two loops $a, b : [0, 1] \to X$ with initial and final point p are said to be *equivalent* if there is a continuous family h_s ($s \in [0, 1]$) of loops in X with initial and final point p, such that $h_0 = a$ and $h_1 = b$. For instance, the figure on the right shows two loops a_1, a_2 which are equivalent to the constant loop e, while a_3, a_4 are equivalent to each other but not to e, since (intuitively) they wind once around the hole in X.

The text proves that the set of equivalence classes of p-based loops forms a group, called the *fundamental group* $\pi(X, p)$ of X relative to basepoint p, with multiplication given by path concatenation, and inverse given by path reversal. Furthermore, if X is path-connected then this group is independent of the choice of basepoint, so we may write $\pi(X)$ for the fundamental group of X.

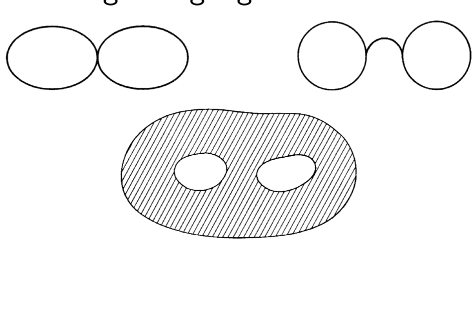
Since we have no other topological tools at our disposal, calculating the fundamental group of a space by hand is quite tedious in general. It is easy to show that convex sets have trivial fundamental group. However, Crowell and Fox spend 5 pages proving that $\pi(S^1) \cong \mathbb{Z}$, using the fact that $\mathbb{R}$ is a covering space for $S^1 \cong \mathbb{R}/\mathbb{Z}$.

Actually, with a bit of complex analysis, we can formulate a shorter proof of the above fact using winding numbers: the key is to show that

$$\frac{1}{2\pi i} \oint_\gamma \frac{dz}{z} \in \mathbb{Z}$$

for any 'well-behaved' (eg. piecewise smooth) loop γ on the unit circle in $\mathbb{C}$, and then using γ to approximate any continuous loop on the unit circle.

Chapters 3 and 4 give an account of free groups and group presentations, which is outside of the scope of this report. However, one notable theorem that is used later on is the Tietze theorem on group presentations, which states that any presentation of a group (in terms of generators and relations) can be converted to any other presentation by a series of elementary operations, each either adding or removing a single generator or relation.



Chapter 5 is concerned with the explicit calculation of fundamental groups. It starts off with some examples of spaces which intuitively should have isomorphic fundamental groups (see figure). To prove this rigorously, the text develops the theory of how the fundamental group behaves under various topological transformations. For instance, a map from X to a subspace Y is called a *retraction* if its restriction to Y is the identity map. A *deformation* of X into Y is a family of maps $h_s : X \to X$ ($s \in [0, 1]$) such that $h_0 = \mathrm{id}_X$, $H(s, p) = h_s(p)$ is continuous in both s and p, and $h_1(X) \subseteq Y$. The main result concerning these transformations is the following:

Proposition. *Let X be path-connected, and let Y be a subspace of X. If X can be deformed into Y and there exists a retraction $X \to Y$, then $\pi(X) \cong \pi(Y)$.*

We say that two topological spaces have the same *homotopy type* if they are related to each other by a finite sequence of homeomorphisms and deformation retractions. Now it is easy to check that the three example spaces at the start of the chapter are all of the same homotopy type, which proves that their fundamental groups are isomorphic. In fact, most invariants that appear in algebraic topology (such as the fundamental group) are invariants of homotopy type, rather than topological equivalence.

Next, we have a modern statement of the van Kampen theorem, essentially as a universal property of the fundamental group of a space with respect to the fundamental group of its parts:

Theorem. *Let X be a topological space, and let X_1, X_2 be open subsets such that $X = X_1 \cup X_2$, and that X_1, X_2 and $X_0 = X_1 \cap X_2$ are path-connected and nonempty. Let $G = \pi(X)$ and $G_i = \pi(X_i)$. If the inclusion maps induce homomorphisms in the following commutative diagram:*

$$\begin{tikzcd} & G_0 \arrow[dl, "\theta_1"'] \arrow[dr, "\theta_2"] \arrow[dd, "\omega_0"] & \\ G_1 \arrow[dr, "\omega_1"'] & & G_2 \arrow[dl, "\omega_2"] \\ & G & \end{tikzcd}$$

Then $\omega_1(G_1)$ and $\omega_2(G_2)$ generate G.

Furthermore, if H is any group, and $\psi_i : G_i \to H$ are homomorphisms such that the following diagram commutes:

$$\begin{tikzcd} & G_0 \arrow[dl, "\theta_1"'] \arrow[dr, "\theta_2"] \arrow[dd, "\psi_0"] & \\ G_1 \arrow[dr, "\psi_1"'] & & G_2 \arrow[dl, "\psi_2"] \\ & H & \end{tikzcd}$$

Then there exists a unique homomorphism $\lambda : G \to H$ such that $\psi_i = \lambda \circ \omega_i$.

This can be seen as the rigorous statement of the intuitive idea that G is the freest possible group that extends G_1 and G_2, subjected to natural restrictions posed by G_0. Hence we obtain G by 'gluing' together the groups G_1 and G_2 'along' the homomorphisms θ_1 and θ_2. The classical statement of van Kampen's theorem, involving group presentations, is derived as a corollary to the above formulation.

With all these new tools, we finally return to knot theory in Chapter 6. The big idea is that for any knot K, $\pi(\mathbb{R}^3 \backslash K)$ *only depends on the knot type of* K. In this chapter, Crowell and Fox describe a standard way of calculating a presentation for this group.

Firstly, we partition a polygonal knot drawing of K into n overpasses (bold lines in figure) and n underpasses (light lines). We can imagine that the bold segments are the parts of the knot above the xy-plane, and the light segments are the parts below.

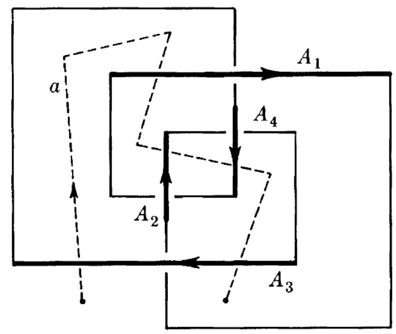
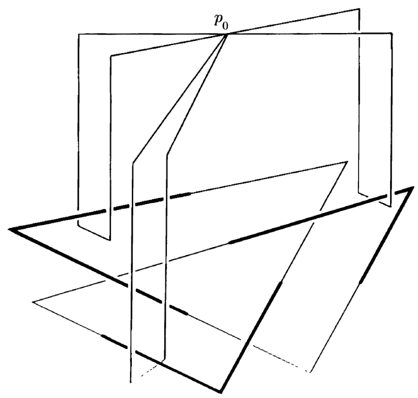
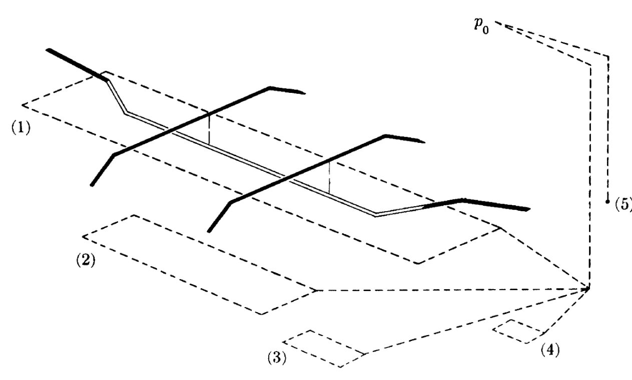


Figure 28. $a^\sharp = x_3x_1x_2x_4^{-1}x_3^{-1}$



Fix a point p_0 high above the xy-plane. Now for any path a on the xy-plane not intersecting the knot K, we can associate a word $a^\sharp$ to the path, based on the (oriented) crossings between the path and the overpasses of the knot. This word then corresponds to a loop based at p_0 (see figure). Hence we can get n generators for the group $\pi(\mathbb{R}^3 \backslash K)$.

Now it turns out that the only nontrivial relations in this group are induced by situations such as in the diagram below, where we pull out the dashed loop from below the underpass strand and contract it to a point. Each such loop gives us one relation for each underpass, so we get n relations in total, and it can be shown that exactly one of these relations is redundant. Hence we have a method to get a presentation for $\pi(\mathbb{R}^3 \backslash K)$ with n generators and $n-1$ relations.



This is sufficient, for instance, to prove that the trefoil knot is not equivalent to the unknot: the trefoil knot group has an homomorphism onto the symmetric group of 3 elements, which is nonabelian, while the unknot group is just $\mathbb{Z}$. However, the word problem (deciding whether two group presentations yield isoorphic groups) is undecidable in general, so the procedure given above is not a computationally feasible method to decide whether two knots are equivalent.

Now we want to extract knot invariants from the knot group which are more suited to the problem of distinguishing knots, ie. easier to calculate. This is the aim of the rest of the book, which presents constructions for more algebraic structures over the knot group, such as the group ring, derivations, elementary ideals, and the knot polynomials. These are beyond the scope of this report. However, we note that Crowell and Fox give an example of a pair of inequivalent knots which have the same fundamental group; hence the algebraic methods presented in this book will sometimes fail to detect distinct knots.

It is interesting to keep track of the developments in knot theory since the publication of the book. The end goal of Crowell and Fox's text is to describe the Alexander polynomial, which is the state-of-the-art in knot invariants at the time. In the late 1960s, John Conway discovered a simple relation between the Alexander polynomials of certain knots, now known as skein relations. This led to further developments in the 1980s, such as the Jones polynomial by Vaughan Jones, culminating in the HOMFLY polynomial which generalises both of the above knot polynomials.

Another research direction in the late 1970s to the 1980s was the introduction of geometry. William Thurston proved that for any knot K, the complement $\mathbb{R}^3 \backslash K$ can be equipped with a hyperbolic geometry. This perspective on the problem yields entirely new knot invariants, such as the hyperbolic volume of the knot complement, a positive real that can be computed to arbitrary precision. This provides new algorithmic approaches to distinguish between different knots.

In summary, we have started from a purely topological question (classifying inequivalent knots), and converted it into a study of algebraic objects, in this case the fundamental group. The problem can then be attacked via algebraic methods, which leads to new insight for the original problem in topology. In my view, it is exactly at such boundaries between different fields—where seemingly unrelated ideas can merge, and provide deeper understanding towards a problem—it is exactly at the boundaries where mathematics is the most alive.

References

R. H. Crowell and R. H. Fox, *Introduction to Knot Theory*, Springer-Verlag, 1963.